Conformal Supergravity Tree Amplitudes from Open Twistor String Theory

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Abstract

We display the vertex operators for all states in the conformal supergravity sector of the twistor string, as outlined by Berkovits and Witten. These include 'dipole' states, which are pairs of supergravitons that do not diagonalize the translation generators. We use canonical quantization of the open string version of Berkovits, and compute N-point tree level scattering amplitudes for gravitons, gluons and scalars. We reproduce the Berkovits-Witten formula for maximal helicity violating (MHV) amplitudes (which they derived using path integrals), and extend their results to the dipole pairs. We compare these trees with those of Einstein gravity field theory.

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1 Introduction

We pursue the tree amplitudes for graviton scattering in conformal gravity, described by twistor string theory. The twistor string [1] and its open string formulation [2] describe massless particles of $\mathcal{N} = 4$ Yang-Mills theory coupled to conformal supergravity [3] in four-dimensional Minkowski spacetime.

Conformal gravity field theories [4, 5] provided early examples of finite field theories of gravity [6, 7]. They are not unitary theories, but have interesting structure and continue to provoke comments about possible uses [8]. Of course the conformal supergraviton states have zero norm, due to the lack of unitarity [9]. Nevertheless, the equivalence of the twistor string with this field theory system can be exploited to derive conformal gravity tree level scattering amplitudes hard to access in the field theory. We compute the gravity trees as a step toward learning how to decouple them in the twistor string. This would result in a perturbative string theory for super Yang-Mills (with no tower of massive states), and the computational advantage one hopes for in a string theory vs. field theory description. Various efforts towards a QCD string are discussed in [10].

We work in a spinor helicity basis [11]-[13], and compare the conformal gravity tree amplitudes with those of Einstein gravity [14]-[19]. The conformal gravity trees have fewer poles. We compute the conformal couplings in detail, as they should be important in further study of the loop calculation [20].

Computation is done in the Berkovits open string version [2]. We use the twistor string canonical quantization described in [20, 21] and follow their notation. In section 2, we give the vertex operators for all states in the conformal supergravity multiplets, as outlined by Berkovits and Witten [3]. These include the dipole states, which form pairs of supergravitons, where one state in each pair does not diagonalize the translation generators, and is not a momentum eigenstate. We show all supergraviton states have zero norm in our basis.

In section 3, three-point scattering amplitudes for gluons and gravitons and scalars, with both one and two negative helicities are calculated. We include cases for both members the dipoles, and find a momentum derivative appearing in amplitudes for states that do not diagonalize the translations generators. These amplitudes still have translational invariance.

In section 4, we extend our results to N-point tree level amplitudes for these dipole pairs. We reproduce the Berkovits-Witten formula for maximal helicity violating (MHV) amplitudes for the diagonal states, showing consistency of the canonical approach and the path integral framework. In section 5, we compute N-point conformal gravity MHV tree amplitudes for a selection of gluons and supergravitons in the dipole pairs.

2 Vertex Operators and Canonical Quantization

The world-sheet fields for twistor string theory are the twistor fields Y_I, Z^I, $1 \leq I \leq 8$, and the current algebra J^A with central charge 28. In addition, there are ghost fields b, c, u, v, and world sheet gauge fields all summarized in [20]. The fields Z^I have conformal spin zero and are relabeled as four boson fields $\lambda^a, \mu^{\dot{a}}$, $1 \leq a, \dot{a} \leq 2$ and four fermion fields ψ^m, $1 \leq m \leq 4$. The conjugate variables Y_I have conformal spin one, as do the currents J^A.

The twistor field commutation relations follow from

$$Z^I(\rho)Y_J(\zeta) =: Z^I(\rho)Y_J(\zeta) : +\delta^I_J(\rho-\zeta)^{-1}. \tag{2.1}$$

2.1 Vertex Operators

The massless states of $\mathcal{N} = 4$ conformal supergravity consist of pairs of graviton supermultiplets (called dipoles), whose vertex operators are $V_F(\rho)$, $V_{F'}(\rho)$ and $V_G(\rho), V_{G'}(\rho)$; in addition to spin 3/2 supermultiplets, with vertex operators $V_f(\rho)$ and $V_g(\rho)$. Loosely following the notation of [3], we list them in terms of homogeneous functions f^I, g_I, of Z^I in Table 1. For each vertex, f^I and g_I satisfy

$$\frac{\partial}{\partial Z^I} f^I = 0, \quad Z^I g_I = 0 \tag{2.2}$$

to ensure the vertex operators are primary with respect to the $U(1)$ current

$$J(\rho) = -\sum_I : Y_I(\rho)Z^I(\rho) : \tag{2.3}$$

and the Virasoro current

$$L(\rho) = -\sum_I : Y_I(\rho)Z^I(\rho) : - : u(\rho)v(\rho) : +2 : \partial c(\rho)b(\rho) : - : \partial b(\rho)c(\rho) : +L^J(\rho). \tag{2.4}$$

Here $L^J(\rho)$ is the contribution from the current algebra. The vertex operators have charge zero and conformal dimension one. The primed vertices correspond to states that do not diagonalize the translation generators [3].

Vertex Operator	Helicities
$V_F(\rho) = f^{\dot{a}}(Z(\rho))Y_{\dot{a}}(\rho)$	$(2,\ \frac{3}{2},\ 1,\ \frac{1}{2},\ 0)$
$V_G(\rho) = g_a(Z(\rho))\partial\lambda^a(\rho)$	$(0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},\ -2)$
$V_{F'}(\rho) = f^a(Z(\rho))Y_a(\rho) + \hat{f}^{\dot{a}}(Z(\rho))Y_{\dot{a}}(\rho)$	$(2,\ \frac{3}{2},\ 1,\ \frac{1}{2},\ 0)$
$V_{G'}(\rho) = g_{\dot{a}}(Z(\rho))\partial\mu^{\dot{a}}(\rho) + \hat{g}_a(Z(\rho))\partial\lambda^a(\rho)$	$(0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2},\ -2)$
$V_f(\rho) = f^m(Z(\rho))Y_m(\rho) + \tilde{f}^{\dot{a}}(Z(\rho))Y_{\dot{a}}(\rho)$	$(\frac{3}{2},\ 1,\ \frac{1}{2},\ 0,\ -\frac{1}{2})$
$V_g(\rho) = g_m(Z(\rho))\partial\psi^m(\rho) + \tilde{g}_a(Z(\rho))\partial\lambda^a(\rho)$	$(\frac{1}{2},\ 0,\ -\frac{1}{2},\ -1,\ -\frac{3}{2})$
$V_\Phi^A(\rho) = V_\phi(Z(\rho))J^A(\rho)$	$(\pm 1,\ 4(\pm\frac{1}{2}),\ 6(0))$

Table 1: Vertex operators and helicities for $\mathcal{N}=4$ conformal supergravity and Yang-Mills theory

We will define the homogeneous functions for each vertex operator, and discuss their properties. The states are labeled by helicities and their representations under the $SU(4)$ $R-$symmetry (in bold). As a reminder, we first look at the $\mathcal{N}=4$ Yang-Mills gluon vertex,

$$V_\Phi^A(\rho) = V_\phi(Z(\rho))J^A(\rho) \tag{2.5}$$

with

$$\begin{aligned}
V_\phi(Z(\rho)) &= \int \frac{dk}{k}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \\
&\quad\times\left[A_1 + k\psi^b A_b + \frac{k^2}{2}\psi^b\psi^c A_{bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d A_{bcd} + k^4\psi^1\psi^2\psi^3\psi^4 A_{-1}\right] \\
&= \frac{1}{(\pi^1)^2}\delta\left(\frac{\lambda^2(\rho)}{\lambda^1(\rho)} - \frac{\pi^2}{\pi^1}\right)\exp\left\{i\frac{\mu^{\dot{b}}(\rho)\bar{\pi}_{\dot{b}}\pi^1}{\lambda^1(\rho)}\right\} \\
&\quad\times\left[A_+ + \frac{\pi^1}{\lambda^1(\rho)}\psi^b(\rho)A_b + \left(\frac{\pi^1}{\lambda^1(\rho)}\right)^2\frac{1}{2}\psi^b(\rho)\psi^c(\rho)A_{bc}\right. \\
&\quad\left. + \left(\frac{\pi^1}{\lambda^1(\rho)}\right)^3\frac{1}{3!}\psi^b(\rho)\psi^c(\rho)\psi^d(\rho)A_{bcd} + \left(\frac{\pi^1}{\lambda^1(\rho)}\right)^4\psi^1(\rho)\psi^2(\rho)\psi^3(\rho)\psi^4(\rho)A_-\right]
\end{aligned} \tag{2.6}$$

where $\psi^b \equiv \psi^b(\rho)$ and b,c,d are summed over. With use of the delta function $\delta(k\lambda^1(\rho)-\pi^1)$ to perform the k-integration, this becomes the vertex used by Berkovits and Witten, except they omit the A_b, A_{bc}, and A_{bcd} terms [20, 2, 22, 3]. In that form, it is easy to see that the

vertex operator $V_\phi(Z^I(\rho))$ is homogeneous in $Z^I(\rho)$ of degree $p=0$. (A function homogeneous in Z of degree p satisfies $f(kZ)=k^p f(Z)$, so it has $U(1)$ charge p.) For the scaling $\pi^a \to \kappa\pi^a$, $\bar{\pi}^a \to \kappa^{-1}\bar{\pi}^a$, each helicity component scales as κ^{-2h} where h is the helicity of the state in Minkowski spacetime [3]. Thus $V_\phi(Z(\rho))$ describes the super gluon helicity states $(1,\mathbf{1}),(\frac{1}{2},\bar{\mathbf{4}}),(0,6),(-\frac{1}{2},\mathbf{4}),(-1,\mathbf{1})$. The spinor helicity variables $\pi^a,\bar{\pi}^{\dot{a}}$ are related to massless four-dimensional momentum $p_{a\dot{a}}=\pi_a\bar{\pi}_{\dot{a}}=\sigma^\mu_{a\dot{a}}p_\mu$ where $\sigma^\mu=(1,\sigma^i)$ in terms of the Pauli matrices σ^i. Indices are raised and lowered $q_a=\epsilon_{ab}q^b$, $q^a=\epsilon^{ab}q_b$, $q_{\dot{a}}=\epsilon_{\dot{a}\dot{b}}q^{\dot{b}}$, $q^{\dot{a}}=\epsilon^{\dot{a}\dot{b}}q_{\dot{b}}$ with $\epsilon^{12}=1=-\epsilon_{12}$.

F Vertices

For the conformal supergravity states, the vertex operator for the helicity states $(2,\mathbf{1}),(\frac{3}{2},\bar{\mathbf{4}}),(1,\mathbf{6}),(\frac{1}{2},\mathbf{4}),(0,\mathbf{1})$ is given by

$$V_F(\rho)=f^{\dot{a}}(Z(\rho))\,Y_{\dot{a}}(\rho) \tag{2.7}$$

with

$$\begin{aligned} f^{\dot{a}}(Z(\rho)) = i\int \frac{dk}{k^2}\bar{\pi}^{\dot{a}}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \\ \times\left[e_2+k\psi^b\eta_{\frac{3}{2}b}+\frac{k^2}{2}\psi^b\psi^c T_{1bc}+\frac{k^3}{3!}\psi^b\psi^c\psi^d\Lambda_{\frac{1}{2}bcd}+k^4\psi^1\psi^2\psi^3\psi^4\bar{C}_0\right]. \end{aligned} \tag{2.8}$$

The function $f^{\dot{a}}(Z^I(\rho))$ is homogeneous in $Z^I(\rho)$ of degree 1. The highest component (which is proportional to e_2) scales as κ^{-4} with π^a and $\bar{\pi}^a$, to describe helicity 2. As required by the primary field conditions, $\frac{\partial}{\partial\mu^{\dot{a}}(\rho)}f^{\dot{a}}(Z(\rho))=0$, since $\bar{\pi}_{\dot{a}}\bar{\pi}^{\dot{a}}=0$. These vertices correspond to plane wave states and diagonalize the translation generators. Together with the F' vertices they comprise a dipole pair [3].

F′ Vertices

The vertex operator for a second set of states $(2,\mathbf{1}),(\frac{3}{2},\bar{\mathbf{4}}),(1,\mathbf{6}),(\frac{1}{2},\mathbf{4}),(0,\mathbf{1})$ is

$$V_{F'}(\rho)=f^a(Z(\rho))\,Y_a(\rho)+\hat{f}^{\dot{a}}(Z(\rho))\,Y_{\dot{a}}(\rho) \tag{2.9}$$

with

$$f^a(Z(\rho)) = \bar{s}^a \int \frac{dk}{k^2} \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \times \left[e'_2 + k\psi^b \eta'_{\frac{3}{2}b} + \frac{k^2}{2}\psi^b\psi^c T'_{1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \Lambda'_{\frac{1}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 \bar{C}'_0 \right] \tag{2.10}$$

and

$$\hat{f}^{\dot{a}}(Z(\rho)) = -is^{\dot{a}}\bar{s}^e \int \frac{dk}{k^3} \frac{\partial}{\partial\lambda^e(\rho)} \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \times \left[e'_2 + k\psi^b \eta'_{\frac{3}{2}b} + \frac{k^2}{2}\psi^b\psi^c T'_{1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \Lambda'_{\frac{1}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 \bar{C}'_0 \right] \tag{2.11}$$

chosen to satisfy the volume preserving condition $\frac{\partial}{\partial\lambda^a(\rho)} f^a(Z(\rho)) + \frac{\partial}{\partial\mu^{\dot{a}}(\rho)} \hat{f}^{\dot{a}}(Z(\rho)) = 0$. The spinors $s_{\dot{a}}$ and $\bar{s}_a$ are defined such that $\pi^a \bar{s}_a = 1$ and $\bar{\pi}^{\dot{a}} s_{\dot{a}} = 1$. These states are not eigenstates of the momentum operator, as we discuss in (3.25).

G Vertices

Conformal supergravity states with the opposite helicities and conjugate $SU(4)$ representations, $(0, \mathbf{1}), (-\frac{1}{2}, \bar{\mathbf{4}}), (-1, \mathbf{6}), (-\frac{3}{2}, \mathbf{4}), (-2, \mathbf{1})$ are described by

$$V_G(\rho) = g_a(Z(\rho))\, \partial\lambda^a(\rho) \tag{2.12}$$

with

$$g_a(Z(\rho)) = \int dk\, k\, \lambda_a(\rho) \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \times \left[C_0 + k\psi^b \Lambda_{-\frac{1}{2}b} + \frac{k^2}{2}\psi^b\psi^c T_{-1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \eta_{-\frac{3}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 e_{-2} \right]. \tag{2.13}$$

$g_a(Z^J(\rho))$ is homogeneous in $Z^J(\rho)$ of degree -1. The highest component (proportional to C), scales with π^a and $\bar{\pi}^a$ as κ^0 for zero helicity. Also, $\lambda^a(\rho) g_a(Z(\rho)) = 0$. These are momentum eigenstates, and form a dipole pair with the G' vertices.

G' Vertices

The final states that do not diagonalize the translation generators form a second set of states $(0,\mathbf{1}),(-\frac{1}{2},\bar{\mathbf{4}}),(-1,\mathbf{6}),(-\frac{3}{2},\mathbf{4}),(-2,\mathbf{1})$ and correspond to

$$V_{G'}(\rho) = g_{\dot{a}}(Z(\rho))\,\partial\mu^{\dot{a}}(\rho) + \hat{g}_a(Z(\rho))\,\partial\lambda^a(\rho) \tag{2.14}$$

with

$$g_{\dot{a}}(Z(\rho)) = is_{\dot{a}}\int dk\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \\ \times\left[C'_0 + k\psi^b\Lambda'_{-\frac{1}{2}b} + \frac{k^2}{2}\psi^b\psi^c T'_{-1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d\eta'_{-\frac{3}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 e'_{-2}\right] \tag{2.15}$$

and

$$\hat{g}_a(Z(\rho)) = -i\bar{s}_a s_{\dot{a}}\mu^{\dot{a}}(\rho)\int dk\,k\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \\ \times\left[C'_0 + k\psi^b\Lambda'_{-\frac{1}{2}b} + \frac{k^2}{2}\psi^b\psi^c T'_{-1bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d\eta'_{-\frac{3}{2}bcd} + k^4\psi^1\psi^2\psi^3\psi^4 e'_{-2}\right] \tag{2.16}$$

with $\mu^{\dot{a}}(\rho)g_{\dot{a}}(Z(\rho)) + \lambda^a(\rho)\hat{g}_a(Z(\rho)) = 0$.

f Vertices

The vertex operator for the plane wave states with quantum numbers

$(\frac{3}{2},\mathbf{4})$, $(1,\mathbf{15}\oplus\mathbf{1})$, $(\frac{1}{2},\,\overline{\mathbf{20}}\oplus\bar{\mathbf{4}})$, $(0,\mathbf{10}\oplus\mathbf{6})$, $(-\frac{1}{2},\mathbf{4})$ is

$$V_f(\rho) = \; f^m\,(Z(\rho))\,Y_m(\rho) \;+\; \tilde{f}^{\dot{a}}\,(Z(\rho))\,Y_{\dot{a}}(\rho) \tag{2.17}$$

with

$$f^m(Z(\rho)) = \int\frac{dk}{k^2}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)} \\ \times\left[E^m_{\frac{3}{2}} + k\psi^b E^m_{1b} + \frac{k^2}{2}\psi^b\psi^c E^m_{\frac{1}{2}bc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d E^m_{0bcd} + k^4\psi^1\psi^2\psi^3\psi^4 E^m_{-\frac{1}{2}}\right] \tag{2.18}$$

and

$$\tilde{f}^{\dot{a}}(Z(\rho)) = -is^{\dot{a}} \int \frac{dk}{k^2} \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times \left[E^m_{1m} + k\psi^c E^m_{\frac{1}{2}mc} + \frac{k^2}{2}\psi^c\psi^d E^m_{0mcd} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \epsilon_{mbcd} E^m_{-\frac{1}{2}} \right] \tag{2.19}$$

so that $\frac{\partial}{\partial\psi^m(\rho)} f^m(Z(\rho)) + \frac{\partial}{\partial\mu^{\dot{a}}(\rho)} \tilde{f}^{\dot{a}}(Z(\rho)) = 0$, $f^m(Z(\rho)$ and $\tilde{f}^{\dot{a}}(Z(\rho))$ have degree 1, and the leading components scale as κ^{-3} and κ^{-2} respectively.

g Vertices

The vertex operator for states with the opposite helicities and conjugate $SU(4)$ representations, $(\frac{1}{2}, \overline{\mathbf{4}})$, $(0, \overline{\mathbf{10}} \oplus \mathbf{6})$, $(-\frac{1}{2}, \mathbf{20} \oplus \mathbf{4})$, $(-1, \mathbf{1} \oplus \mathbf{15})$, $(-\frac{3}{2}, \overline{\mathbf{4}})$ is

$$V_g(\rho) = g_m\left(Z(\rho)\right) \partial\psi^m(\rho) \;+\; \tilde{g}_a\left(Z(\rho)\right) \partial\lambda^a(\rho) \tag{2.20}$$

with

$$g_m(Z(\rho)) = \int dk \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times \left[\bar{E}_{\frac{1}{2}m} + k\psi^b \bar{E}_{0mb} + \frac{k^2}{2}\psi^b\psi^c \bar{E}_{-\frac{1}{2}mbc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \bar{E}_{-1mbcd} + k^4\psi^1\psi^2\psi^3\psi^4 \bar{E}_{-\frac{3}{2}m} \right] \tag{2.21}$$

and

$$\tilde{g}_a(Z(\rho)) = \bar{s}_a \int dk\, k \prod_{a=1}^{2} \delta(k\lambda^a(\rho) - \pi^a) e^{ik\bar{\pi}_{\dot{b}}\mu^{\dot{b}}(\rho)}$$
$$\times\, \psi^m \left[\bar{E}_{\frac{1}{2}m} + k\psi^b \bar{E}_{0mb} + \frac{k^2}{2}\psi^b\psi^c \bar{E}_{-\frac{1}{2}mbc} + \frac{k^3}{3!}\psi^b\psi^c\psi^d \bar{E}_{-1mbcd} \right] \tag{2.22}$$

where $\psi^m(\rho) g_m(Z(\rho)) + \lambda^a(\rho)\tilde{g}_a(Z(\rho)) = 0$. To obtain $\tilde{g}_a(Z(\rho))$, we use $\pi^a \bar{s}_a = 1$ which can be written as $\frac{\pi 1}{\lambda^1(\rho)}\lambda^a(\rho)\bar{s}_a = 1$ on the support of the delta function $\delta\left(\frac{\lambda^2(\rho)}{\lambda^1(\rho)} - \frac{\pi^2}{\pi^1}\right)$.

2.2 *Norm of the States*

We can check that the norms of the one particle supergraviton states are zero. The physical states corresponding to the vertex operators are shown in Table 2,

V_F	$f^{\dot{a}}(Z_0)Y_{\dot{a}(-1)}\|0\rangle$
V_G	$g_a(Z_0)\lambda^a_{(-1)}\|0\rangle$
$V_{F'}$	$\left(f^a(Z_0)Y_{a(-1)} + \hat{f}^{\dot{a}}(Z_0)Y_{\dot{a}(-1)}\right)\|0\rangle$
$V_{G'}$	$\left(g_{\dot{a}}(Z_0)\mu^{\dot{a}}_{(-1)} + \hat{g}_a(Z_0)\lambda^a_{(-1)}\right)\|0\rangle$
V_f	$\left(f^m(Z_0)Y_{m(-1)} + \tilde{f}^{\dot{a}}(Z_0)Y_{\dot{a}(-1)}\right)\|0\rangle$
V_g	$\left(g_m(Z_0)\psi^m_{(-1)} + \tilde{g}_a(Z_0)\lambda^a_{(-1)}\right)\|0\rangle$
V^A_Φ	$V_\phi(Z_0)J^A_{(-1)}\|0\rangle$

Table 2: One particle states

where the mode expansion for the twistor fields is $Z^I(\rho) = \sum_n Z^I_n \rho^{-n},\ Y_J(\rho) = \sum_n Y_{Jn}\rho^{-n-1}$, and the modes annihilating the vacuum are $Z^I_n|0\rangle = 0,\ n \geq 1$, and $Y_{nI}|0\rangle = 0,\ n \geq 0$. The canonical commutation relations are

$$\left[Z^I_n, Y_{Jm}\right] = \delta^I_J\,\delta_{n,-m}, \tag{2.23}$$

and the hermitian conjugates [20] are $(Z^I_n)^\dagger = Z^I_{-n}$, for $1 \leq I \leq 8$; and $(Y^J_n)^\dagger = -Y^J_{-n}$, for $1 \leq J \leq 4$; and $(Y^J_n)^\dagger = Y^J_{-n}$, for $5 \leq J \leq 8$. We find that the only non-vanishing inner products for the conformal graviton states are $\langle 0|V^\dagger_{G'}(0)V_F(0)|0\rangle$, $\langle 0|V^\dagger_{F'}(0)V_G(0)|0\rangle$, and $\langle 0|V^\dagger_f(0)V_g(0)|0\rangle$, so the norms of the supergraviton states vanish in the basis chosen in Table 2. We compute the inner products as follows. For example,

$$\langle 0|V^\dagger_G(0)V_F(0)|0\rangle = \langle 0|\lambda^a_1 g^*_a(Z_0)f^{\dot{a}}(Z_0)Y_{\dot{a}(-1)}|0\rangle = 0, \tag{2.24}$$

since Z^I_0 and $Y_{J(-1)}$ commute, and the $Y_{J(-1)}$ acting to the left annihilate the vacuum. In contrast, the gluon norm is positive,

$$\begin{aligned} &||V_\phi(Z_0)J^A_{-1}|0\rangle|| = \langle 0|J^A_1 V^*_\phi(Z_0)V_\phi(Z_0)J^A_{-1}|0\rangle \\ &= \langle 0|J^A_1 J^A_{-1}|0\rangle \int dZ_0|V_\phi(Z_0)|^2 = k\int dZ_0|V_\phi(Z_0)|^2 > 0, \end{aligned} \tag{2.25}$$

where k is the level of the current algebra, $J^A_n J^B_m = if^{AB}{}_C J^C_{n+m} + kn\delta_{n,-m}\delta^{AB}$.

2.3 *A Subset of Vertex Operators*

We will focus on amplitudes involving a subset of the vertex operators, relabeled in Table 3.

V_F	$e_2(\rho) = i\int dk\, k^{-2}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}\,\bar\pi^{\dot a}Y_{\dot a}(\rho)\,e_2$
	$\bar C(\rho) = i\int dk\, k^{-2}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}\,k^4\psi^1\psi^2\psi^3\psi^4\,\bar\pi^{\dot a}Y_{\dot a}(\rho)\,\bar C_0$
V_G	$C(\rho) = \int dk\, k\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}\,\lambda_a(\rho)\partial\lambda^a(\rho)\,C_0$
	$e_{-2}(\rho) = \int dk\, k\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}\,k^4\psi^1\psi^2\psi^3\psi^4\,\lambda_a(\rho)\partial\lambda^a(\rho)\,e_{-2}$
$V_{F'}$	$e'_2(\rho) = \int dk\, k^{-2}\Big[\bar s^a Y_a(\rho)\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a) + i\bar s^b\big(\frac{\partial}{\partial\pi^b}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)\big)s^{\dot a}Y_{\dot a}(\rho)\Big]\,e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}e'_2$
	$\bar C'(\rho) = \int dk\, k^{-2}\Big[\bar s^a Y_a(\rho)\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a) + i\bar s^b\big(\frac{\partial}{\partial\pi^b}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)\big)s^{\dot a}Y_{\dot a}(\rho)\Big]\,e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}k^4\psi^1\psi^2\psi^3\psi^4\,\bar C'_0$
$V_{G'}$	$C'(\rho) = i\int dk\, k\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)\,\big[k^{-1}s_{\dot a}\partial\mu^{\dot a}(\rho)-s_{\dot a}\mu^{\dot a}(\rho)\bar s_a\partial\lambda^a(\rho)\big]\,e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}C'_0$
	$e'_{-2}(\rho) = i\int dk\, k\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)\,\big[k^{-1}s_{\dot a}\partial\mu^{\dot a}(\rho)-s_{\dot a}\mu^{\dot a}(\rho)\bar s_a\partial\lambda^a(\rho)\big] \times e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}\,k^4\psi^1\psi^2\psi^3\psi^4\,e'_{-2}$
V_Φ	$A^A_1(\rho) = \int dk\, k^{-1}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}\,A_1\;J^A(\rho)$
	$A^A_{-1}(\rho) = \int dk\, k^{-1}\prod_{a=1}^{2}\delta(k\lambda^a(\rho)-\pi^a)e^{ik\bar\pi_{\dot b}\mu^{\dot b}(\rho)}k^4\psi^1\psi^2\psi^3\psi^4\,A_{-1}\,J^A(\rho)$

Table 3: A subset of the vertex operators: for conformal gravitons, scalars and gluons

3 Three-Point Couplings

In this section, we compute the non-vanishing three-point amplitudes for the gravitons, scalars, and gluons in the V_F, V_G and V_Φ vertices using canonical quantization, and then extend these to the corresponding states in the primed vertices, $V_{F'}$, and $V_{G'}$. We have relabeled this subset of vertex operators for positive and negative helicity states in Table 3. (It will be convenient to consider the scalars $\bar{C}, \bar{C}'$ as negative helicity, and C, C' as positive helicity when computing amplitudes, as in [3].) Amplitudes for other states can be calculated with similar ease.

Scattering amplitudes in twistor string theory receive contributions from the various instanton sectors, which are due to world sheet gauge fields [1, 2]. Amplitudes with the number of negative helicity states equal to $d+1-\ell$, are computed with instanton number d, where ℓ is the number of loops. For tree amplitudes, $\ell = 0$. We compute the N-point tree as [20]

$$\langle V_1(\rho_1)V_2(\rho_2)\ldots V_N(\rho_N)\rangle_{\text{tree}} = \int \langle 0|e^{dq_0} V_1(\rho_1)V_2(\rho_2)\cdots V_N(\rho_N)|0\rangle \prod_{r=1}^{N} d\rho_r/d\gamma_M d\gamma_S \quad (3.1)$$

where $d\gamma_M$ is the invariant measure of the Mobius group, and $d\gamma_S$ is the invariant measure of the scaling group. q_0 is the conjugate zero mode of the $U(1)$ current and commutes with field modes as $Y^I_{n-d}e^{dq_0} = e^{dq_0}Y^I_n$ and $Z^I_{n+d}e^{dq_0} = e^{dq_0}Z^I_n$.

3.1 *Unprimed Couplings*

Using the canonical methods of [20], we compute the non-vanishing three-point tree amplitudes that come from the degree one curves as follows.

$$\begin{aligned}
\langle A^{A_1}_{-1}(\rho_1)A^{A_2}_{-1}(\rho_2)C(\rho_3)\rangle_{\text{tree}} &= \int \langle 0|e^{q_0} A^{A_1}_{-1}(\rho_1)A^{A_2}_{-1}(\rho_2)C(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= \int \prod_{r=1}^{3} dk_r k_r \lambda_a(\rho_3)\partial\lambda^a(\rho_3) \prod_{r,a} \delta({\pi_r}^a - k_r\lambda^a(\rho_r))(\rho_1-\rho_2)^4(k_1k_2)^4 \\
&\quad \times \langle 0|e^{q_0} e^{i\sum_{r=1}^3 k_r \bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_M d\gamma_S \left(\frac{-\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}C_{0(3)}}{(\rho_1-\rho_2)^2k_1^2k_2^2}\right) \\
&= -\delta^4(\Sigma\pi_r\bar{\pi}_r) \int \prod_{r=1}^{3} d\zeta_r \prod_{r=1}^{3} {\pi_r}^1 \delta({\pi_r}^2 - \zeta_r{\pi_r}^1)(\zeta_1-\zeta_2)^2({\pi_1}^1{\pi_2}^1)^2\, \delta^{A_1A_2}A_{-1(1)}A_{-1(2)}C_{0(3)} \\
&= -\delta^4(\Sigma\pi_r\bar{\pi}_r)\langle 12\rangle^2\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}C_{0(3)} = \epsilon_1^- \cdot p_2\, \epsilon_2^- \cdot p_1\, \delta^4(\Sigma\pi_r\bar{\pi}_r)\delta^{A_1A_2}C_{0(3)}.
\end{aligned} \quad (3.2)$$

This amplitude is type $\phi\phi G$. We replace $Z^I(\rho)$ with $Z_0^I + \rho Z_{-1}^I$, as the only surviving modes, and change variables $\zeta_r = \frac{\lambda^2(\rho_r)}{\lambda^1(\rho_r)}$. For $d = 1$, the invariant measures are expressed as $d\gamma_M d\gamma_S = \prod_{a=1}^2 d^2\lambda^a (\det\lambda)^{-2}$ where $\prod_a d^2\lambda^a = d\lambda_0^1 d\lambda_{-1}^1 d\lambda_0^2 d\lambda_{-1}^2$, and $\det\lambda = \lambda_0^1\lambda_{-1}^2 - \lambda_0^2\lambda_{-1}^1$. The current algebra contribution follows from (4.5). The gluon polarizations are given by $\epsilon_r^- = A_{-1(r)}\pi_{ra}s_{r\dot{a}}$ and $\epsilon_r^+ = A_{1(r)}\bar{s}_{ra}\bar{\pi}_{r\dot{a}}$. We use momentum conservation and thus $s_{s\dot{b}}\sum_r \pi_r^b\bar{\pi}_r^{\dot{b}} = 0$ to find $s_{1\dot{b}}\bar{\pi}_2^{\dot{b}} = \frac{\langle 31\rangle}{\langle 23\rangle}$ and $s_{2\dot{b}}\bar{\pi}_1^{\dot{b}} = \frac{\langle 23\rangle}{\langle 31\rangle}$, so that $\epsilon_1^- \cdot p_2\, \epsilon_2^- \cdot p_1 = -\langle 12\rangle^2\ A_{-1(1)}A_{-1(2)}$, with $\langle rs\rangle = \pi_{ra}\pi_s^a$ and $[rs] = \bar{\pi}_{r\dot{a}}\bar{\pi}^{\dot{a}}$. We can set the scalar wave function $C_{0(3)} = 1$. The momentum conserving delta functions are

$$\prod_{\dot{a},b}\delta\left(\Sigma_{r=1}^3 \pi_r^b\bar{\pi}_{r\dot{a}}\right) \equiv \delta^4(\Sigma\pi_r\bar{\pi}_r). \tag{3.3}$$

For two gluons and a graviton ($\phi\phi G$),

$$\begin{aligned}
&\langle A_1^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)e_{-2}(\rho_3)\rangle_{\text{tree}} = \int \langle 0|e^{q_0}A_1^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)e_{-2}(\rho_3)|0\rangle \prod_{r=1}^3 d\rho_r/d\gamma_M d\gamma_S \\
&= -\int\prod_{r=1}^3 dk_r \prod_{r=1}^3 k_r\ \lambda_a(\rho_3)\partial\lambda^a(\rho_3)\prod_{r,a}\delta({\pi_r}^a - k_r\lambda^a(\rho_r))\ (\rho_2-\rho_3)^4(k_2k_3)^4 \\
&\qquad \times \langle 0|e^{q_0}e^{\sum_{r=1}^3 ik_r{\bar{\pi}}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M \left(\frac{\delta^{A_1A_2}A_{1(1)}A_{-1(2)}e_{-2(3)}}{(\rho_1-\rho_2)^2(k_1k_2)^2}\right) \\
&= -\delta^4(\Sigma\pi_r\bar{\pi}_r)\ \int\prod_{r=1}^3 d\zeta_r\prod_{r=1}^3 \pi_r^1\,\delta(\pi_r^2-\zeta_r\pi_r^1)(\zeta_2-\zeta_3)^4(\pi_2^1\pi_3^1)^4(\zeta_1-\zeta_2)^{-2}(\pi_1^1\pi_2^1)^{-2} \\
&\qquad \times \delta^{A_1A_2}A_{1(1)}A_{-1(2)}e_{-2(3)} \\
&= -\delta^4(\Sigma\pi_r\bar{\pi}_r)\,\frac{\langle 23\rangle^4}{\langle 12\rangle^2}\ \delta^{A_1A_2}A_{1(1)}A_{-1(2)}e_{-2(3)} \\
&= \left(\epsilon_1^+\cdot\epsilon_2^-\ \epsilon^-_{3\,a\dot{a}b\dot{b}}p_1^{a\dot{a}}p_2^{b\dot{b}} + \epsilon_1^+\cdot p_2\ \epsilon^-_{3\,a\dot{a}b\dot{b}}\epsilon_2^{-\,a\dot{a}}p_2^{b\dot{b}} + \epsilon_2^-\cdot p_3\ \epsilon^-_{3\,a\dot{a}b\dot{b}}\epsilon_1^{+\,a\dot{a}}p_2^{b\dot{b}}\right)\delta^{A_1A_2}\ \delta^4(\Sigma\pi_r\bar{\pi}_r).
\end{aligned} \tag{3.4}$$

The gravity polarizations are $\epsilon_r^- = e_{-2(r)}\pi_{ra}s_{r\dot{a}}\pi_{rb}s_{r\dot{b}}$ and $\epsilon_r^+ = e_{2(r)}\bar{s}_{ra}\bar{\pi}_{r\dot{a}}\bar{s}_{rb}\bar{\pi}_{r\dot{b}}$, and one can factor

$$\begin{aligned}
&\epsilon_1^+\cdot\epsilon_2^-\ \epsilon^-_{3\,a\dot{a}b\dot{b}}p_1^{a\dot{a}}p_2^{b\dot{b}} + \epsilon_1^+\cdot p_2\ \epsilon^-_{3\,a\dot{a}b\dot{b}}\epsilon_2^{-\,a\dot{a}}p_2^{b\dot{b}} + \epsilon_2^-\cdot p_3\ \epsilon^-_{3\,a\dot{a}b\dot{b}}\epsilon_1^{+\,a\dot{a}}p_2^{b\dot{b}} \\
&= \left(\epsilon_1^+\cdot\epsilon_2^-\ \epsilon_3^-\cdot p_1 + \epsilon_1^+\cdot p_2\ \epsilon_3^-\cdot\epsilon_2^- + \epsilon_2^-\cdot p_3\ \epsilon_3^-\cdot\epsilon_1^+\right)\epsilon_3^-\cdot p_2 = \frac{\langle 23\rangle^3}{\langle 12\rangle\langle 31\rangle}\,\frac{\langle 23\rangle\langle 31\rangle}{\langle 12\rangle} = \frac{\langle 23\rangle^4}{\langle 12\rangle^2}.
\end{aligned} \tag{3.5}$$

For two gravitons and a scalar (GGG),

$$\langle e_{-2}(\rho_1)e_{-2}(\rho_2)C(\rho_3)\rangle_{\text{tree}} = \int \langle 0|e^{q_0} e_{-2}(\rho_1)e_{-2}(\rho_2)C(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S$$

$$= \int \prod_{r=1}^{3} dk_r \prod_{r=1}^{3} k_r \; \lambda_a(\rho_1)\partial\lambda^a(\rho_1)\, \lambda_b(\rho_2)\partial\lambda^b(\rho_2)\, \lambda_c(\rho_3)\partial\lambda^c(\rho_3) \prod_{ra} \delta(\pi_r^a - k_r\lambda^a(\rho_r))\; (\rho_1-\rho_2)^4 k_1^4 k_2^4$$

$$\times \langle 0|e^{q_0} e^{\sum_{r=1}^{3} ik_r\bar\pi_{r\dot b}\mu^{\dot b}(\rho_r)}|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M\; e_{-2(1)}e_{-2(2)}C_{0(3)}$$

$$= \delta^4(\Sigma\pi_r\bar\pi_r) \int \prod_{r=1}^{3} d\zeta_r \prod_{r=1}^{3} \pi_r^1\, \delta(\pi_r^2 - \zeta_r\pi_r^1)\, (\zeta_1-\zeta_2)^2(\pi_1^1\pi_2^1)^2\; e_{-2(1)}e_{-2(2)}C_{0(3)}$$

$$= \delta^4(\Sigma\pi_r\bar\pi_r)\, \langle 12\rangle^4 e_{-2(1)}e_{-2(2)}C_{0(3)} = \epsilon^-_{1\,a\dot a b\dot b} p_2^{a\dot a} p_2^{b\dot b}\, \epsilon^-_{2\,c\dot c d\dot d} p_1^{c\dot c} p_1^{d\dot d}\;\; \delta^4(\Sigma\pi_r\bar\pi_r)\;\; C_{0(3)}. \tag{3.6}$$

Less conventional is the three-graviton coupling (GGF):

$$\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)\rangle_{\text{tree}} = \int \langle 0|e^{q_0} e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S$$

$$= i\int \prod_{r=1}^{3} dk_r \frac{k_1k_2}{k_3^2}\; \lambda_a(\rho_1)\partial\lambda^a(\rho_1)\, \lambda_a(\rho_2)\partial\lambda^a(\rho_2) \prod_{ra} \delta(\pi_r^a - k_r\lambda^a(\rho_r))\; (\rho_1-\rho_2)^4 k_1^4 k_2^4$$

$$\times \langle 0|e^{q_0} e^{\sum_{r=1}^{3} ik_r\bar\pi_{r\dot b}\mu^{\dot b}(\rho_r)}\; \bar\pi_3^{\dot a} Y_{\dot a}(\rho_3)|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M e_{-2(1)}e_{-2(2)}e_{2(3)}$$

$$= i\delta^4(\Sigma\pi_r\bar\pi_r) \int \prod_{r=1}^{3} d\zeta_r \prod_{r=1}^{3} \delta(\pi_r^2 - \zeta_r\pi_r^1)\, (\zeta_1-\zeta_2)^4(\pi_1^1\pi_2^1)^4\, \frac{\pi_1^1\pi_2^1}{(\pi_3^1)^2}\, (-i) \sum_{r=1}^{2} \frac{\pi_r^1[3r]}{\zeta_r-\zeta_3}\;\; e_{-2(1)}e_{-2(2)}e_{2(3)}$$

$$= \delta^4(\Sigma\pi_r\bar\pi_r)\, \langle 12\rangle^4 \sum_{r=1}^{2} \frac{[3r]\langle r\xi\rangle^2}{\langle 3r\rangle\langle 3\xi\rangle^2}\, e_{-2(1)}e_{-2(2)}e_{2(3)}$$

$$= \delta^4(\Sigma\pi_r\bar\pi_r)\, \langle 12\rangle^6\; \frac{[32]}{\langle 32\rangle\langle 31\rangle^2}\, e_{-2(1)}e_{-2(2)}e_{2(3)} = 0, \tag{3.7}$$

since $\langle 12\rangle[23] = 0$ by momentum conservation. Since this amplitude involves $Y_{\dot a}$, we have first evaluated, using (2.23),

$$\langle 0|e^{q_0} e^{\sum_{r=1}^{3} ik_r\bar\pi_{r\dot b}\mu^{\dot b}(\rho_r)}\, \bar\pi_3^{\dot a} Y_{\dot a}(\rho_3)|0\rangle = -i\sum_{r\neq 3} \frac{k_r[3r]}{(\rho_r-\rho_3)} \langle 0|e^{q_0} e^{\sum_{r=1}^{3} ik_r\bar\pi_{r\dot b}\mu^{\dot b}(\rho_r)}|0\rangle, \tag{3.8}$$

then replaced $\mu^{\dot b}(\rho)$ by its lowest modes and changed variables from ρ_r to ζ_r, as discussed in more detail in (4.8). The expression is independent of the spinor ξ. We compare this

vanishing three-graviton tree amplitude for conformal gravity with that of Einstein gravity,

$$\begin{aligned}\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)\rangle_{Einstein\,tree} &= \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,e_{-2(1)}e_{-2(2)}e_{2(3)} \neq 0 \\ &= \frac{1}{s_{23}}\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)\rangle_{\rm tree}.\end{aligned} \tag{3.9}$$

For two scalars and a graviton (GGF), the amplitude also vanishes by momentum conservation:

$$\begin{aligned}&\langle C(\rho_1)e_{-2}(\rho_2)\bar{C}(\rho_3)\rangle_{\rm tree} = \int \langle 0|e^{q_0}C(\rho_1)e_{-2}(\rho_2)\bar{C}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\ &= i\int \prod_{r=1}^{3} dk_r \frac{k_1k_2}{k_3^2}\,\lambda_a(\rho_1)\partial\lambda^a(\rho_1)\,\lambda_a(\rho_2)\partial\lambda^a(\rho_2)\prod_{ra}\delta(\pi_r^a - k_r\lambda^a(\rho_r))\,(\rho_2-\rho_3)^4k_2^4k_3^4 \\ &\qquad \times \langle 0|e^{q_0}e^{\sum_{r=1}^{3} ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}\,\bar{\pi}_3^{\dot{a}}Y_{\dot{a}}(\rho_3)|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M \quad C_{0(1)}e_{-2(2)}\bar{C}_{0(3)} \\ &= \delta^4(\Sigma\pi_r\bar{\pi}_r)\,\langle 12\rangle^2\,\frac{[23]\langle 23\rangle^3}{\langle 31\rangle^2}C_{0(1)}e_{-2(2)}\bar{C}_{0(3)} = 0.\end{aligned} \tag{3.10}$$

The remaining three-point functions with two negative helicity states also vanish. For comparison, we include the familiar degree one three-point gluon vertex,

$$\begin{aligned}&\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)A_1^{A_3}(\rho_3)\rangle_{\rm tree} = \int \langle 0|e^{q_0}A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)A_1^{A_3}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\ &\quad = \int \prod_{r=1}^{3}\frac{dk_r}{k_r}\prod_{ra}\delta(\pi_r^a - k_r\lambda^a(\rho_r))\,(\rho_1-\rho_2)^4k_1^4k_2^4\,\frac{f^{A_1A_2A_3}}{(\rho_1-\rho_2)(\rho_2-\rho_3)(\rho_3-\rho_1)} \\ &\qquad\qquad \times \langle 0|e^{q_0}e^{\sum_{r=1}^{3} ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}|0\rangle \prod_a d^2\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M\, A_{-1(1)}A_{-1(2)}A_{1(3)} \\ &\quad = \delta^4(\Sigma\pi_r\bar{\pi}_r)\,\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\,f^{A_1A_2A_3}\,A_{-1(1)}A_{-1(2)}A_{1(3)} \\ &\quad = \delta^4(\Sigma\pi_r\bar{\pi}_r)\,f^{A_1A_2A_3}\,\left(\epsilon_1^-\cdot\epsilon_2^-\epsilon_3^+\cdot p_1 + \epsilon_2^-\cdot\epsilon_3^+\epsilon_1^-\cdot p_2 + \epsilon_3^+\cdot\epsilon_1^-\epsilon_2^-\cdot p_3\right).\end{aligned} \tag{3.11}$$

The unprimed MHV three-point functions are summarized in Table 4, where we include their polarizations and momentum conserving delta function, in order to compare with primed couplings in Table 6. Our calculations agree with the general Berkovits-Witten formula [3], derived from path integral methods, and our analysis is useful in extending to the dipole states, and in comparing with conventional field theory couplings.

$\langle A_{-1}^{A_1} A_{-1}^{A_2} C\rangle = -\langle 12\rangle^2\, \delta^{A_1A_2} A_{-1(1)} A_{-1(2)} C_{0(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r)$
$\langle A_{1}^{A_1} A_{-1}^{A_2} e_{-2}\rangle = -\frac{\langle 23\rangle^4}{\langle 12\rangle^2} \delta^{A_1A_2} A_{1(1)} A_{-1(2)} e_{-2(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r)$
$\langle e_{-2} e_{-2} C\rangle = \langle 12\rangle^4 e_{-2(1)} e_{-2(2)} C_{0(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r)$
$\langle e_{-2} e_{-2} e_2\rangle = \frac{\langle 12\rangle^6 [23]}{\langle 23\rangle\langle 31\rangle^2} e_{-2(1)} e_{-2(2)} e_{2(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r) = 0$
$\langle C e_{-2} \bar{C}\rangle = \frac{\langle 12\rangle^2\langle 23\rangle^3 [23]}{\langle 31\rangle^2} C_{0(1)} e_{-2(2)} \bar{C}_{0(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r) = 0$
$\langle A_{-1}^{A_1} A_{-1}^{A_2} A_{1}^{A_3}\rangle = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle} f^{A_1A_2A_3} A_{-1(1)} A_{-1(2)} A_{1(3)} \delta^4(\Sigma\pi_r\bar{\pi}_r)$

Table 4: Unprimed conformal supergravity MHV couplings

We compare these couplings with those of opposite helicities, with instanton number zero:

$$\begin{aligned}
\langle A_1^{A_1}(\rho_1) A_1^{A_2}(\rho_2) \bar{C}(\rho_3)\rangle_{\text{tree}} &= \int \langle 0| A_1^{A_1}(\rho_1) A_1^{A_2}(\rho_2) \bar{C}(\rho_3) |0\rangle \prod_{r=1}^{3} d\rho_r / d\gamma_M d\gamma_S \\
&= i \int \prod_{r=1}^{3} dk_r \frac{k_3^2}{k_1 k_2} \prod_{r,a} \delta({\pi_r}^a - k_r \lambda^a) \langle 0| e^{\sum_{r=1}^3 i k_r \bar{\pi}_{r\dot{b}} \mu^{\dot{b}}(\rho_r)}\; \bar{\pi}_3^{\dot{a}} Y_{\dot{a}}(\rho_3)\, |0\rangle \\
&\qquad \times \prod_a d\lambda^a \prod_r d\rho_r / d\gamma_S d\gamma_M \left(- \frac{\delta^{A_1A_2} A_{1(1)} A_{1(2)} \bar{C}_{0(3)}}{(\rho_1 - \rho_2)^2} \right) \\
&= - \int \prod_{r=1}^{3} dk_r \frac{k_3^2}{k_1 k_2} \prod_{ra} \delta(\pi_r^a - k_r \lambda^a) \left(\sum_{r \neq 3} \frac{k_r [3r]}{(\rho_r - \rho_3)} \right) \prod_{a=1}^{2} \delta(\sum_{r=1}^{3} k_r \bar{\pi}_{ra}) \\
&\qquad \times \prod_a d\lambda^a \prod_r d\rho_r / d\gamma_S d\gamma_M \quad \delta^{a_1a_2} A_{1(1)} A_{1(2)} \bar{C}_{0(3)}\, (\rho_1 - \rho_2)^{-2} \\
&= -\delta^{a_1a_2} [31] \frac{(\pi_3^1)^2}{\pi_2^1} \int \frac{d\lambda^2}{\lambda^1} \prod_{r=1}^{3} \delta(\pi_r^2 - \frac{\lambda^2}{\lambda^1} \pi_r^1) \prod_a \delta(\sum_{r=1}^{3} \frac{\pi_r^1}{\lambda^1} \bar{\pi}_{ra})\; A_{1(1)} A_{1(2)} \bar{C}_{0(3)} \\
&= -\,\delta^{A_1A_2}\; \delta^4(\Sigma\pi_r\bar{\pi}_r)\; [12]^2\; A_{1(1)} A_{1(2)} \bar{C}_{0(3)}.
\end{aligned} \tag{3.12}$$

After eliminating $Y_{\dot{b}}$, for degree $d = 0$ we replace $Z^I(\rho)$ with Z_0^I. From momentum conservation, we find $\pi_2^1[21] = -\pi_3^1[31]$. We can replace two of the delta functions $\prod_{r=2}^3 \delta(\pi_r^2 - \frac{\lambda^2}{\lambda^1}\pi_r^1)$ with $\prod_{a=1}^2 \delta(\sum_{r=1}^3 (\pi_r^2 - \frac{\lambda^2}{\lambda^1}\pi_r^1)\bar{\pi}_{ra})\, [23] = \prod_{a=1}^2 \delta(\sum_{r=1}^3 \pi_r^2 \bar{\pi}_{ra})\; [23]$. Here $\langle 0|\psi_0^1|0\rangle = 1$, see Ref. [20]. This amplitude is type $\phi\phi F$. It is useful to express the invariant measures as

$$d\gamma_M = \prod_{r=1}^{3} d\rho_r \frac{1}{(\rho_1 - \rho_2)(\rho_2 - \rho_3)(\rho_3 - \rho_1)} \qquad \text{and} \qquad d\gamma_S = \frac{d\lambda^1}{\lambda^1} \tag{3.13}$$

in the $d = 0$ sector, where $\lambda^a = \lambda^a_0$. Comparing (3.12) with (3.2), we verify the $d = 0$ tree $\langle A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)\bar{C}(\rho_3)\rangle_{\rm tree}$ is the antiholomorphic version of the $d = 1$ coupling, $\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C(\rho_3)\rangle_{\rm tree}$. Similarly, the $\phi\phi F$ tree

$$
\begin{aligned}
\langle A_{-1}^{A_1}(\rho_1)A_1^{A_2}(\rho_2)e_2(\rho_3)\rangle_{\rm tree} &= \int \langle 0|A_{-1}^{A_1}(\rho_1)A_1^{A_2}(\rho_2)e_2(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= i\int \prod_{r=1}^{3} dk_r\, \frac{k_1^3}{k_2k_3^2} \prod_{ra}\delta(\pi_r^a - k_r\lambda^a)\langle 0|e^{\sum_{r=1}^3 ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}\bar{\pi}_3^{\dot{a}}Y_{\dot{a}}(\rho_3)|0\rangle \\
&\qquad \times \prod_a d\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M \left(-\frac{\delta^{A_1A_2}A_{-1(1)}A_{1(2)}e_{2(3)}}{(\rho_1-\rho_2)^2}\right) \\
&= -\delta^{A_1A_2}[31]\frac{(\pi_3{}^1)^2}{\pi_2{}^1}\int \frac{d\lambda^2}{\lambda^1}\prod_{r=1}^{3}\delta(\pi_r{}^2 - \frac{\lambda^2}{\lambda^1}\pi_r{}^1)\, \prod_a \delta(\sum_{r=1}^{3}\frac{\pi_r^1}{\lambda^1}\bar{\pi}_{ra})\, A_{-1(1)}A_{1(2)}e_{2(3)} \\
&= -\,\delta^{A_1A_2}\,\delta^4(\Sigma\pi_r\bar{\pi}_r)\,\frac{[23]^4}{[12]^2}\,A_{-1(1)}A_{1(2)}e_{2(3)},
\end{aligned}
\tag{3.14}
$$

is the antiholomorphic version of (3.4). The FFF amplitude

$$
\begin{aligned}
\langle e_2(\rho_1)e_2(\rho_2)\bar{C}(\rho_3)\rangle_{\rm tree} &= \int \langle 0|e_2(\rho_1)e_2(\rho_2)\bar{C}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= -i\int \prod_{r=1}^{3} dk_r\, \frac{k_3^2}{k_1^2k_2^2}\prod_{ra}\delta(\pi_r{}^a - k_r\lambda^a)\,\langle 0|\prod_{r=1}^{3} e^{ik_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_r)}\,\bar{\pi}_r^{\dot{a}}Y_{\dot{a}}(\rho_r)|0\rangle \\
&\qquad \times \prod_a d\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M\, e_{2(1)}e_{2(2)}\bar{C}_{0(3)} \\
&= -i\int \prod_{r=1}^{3} dk_r\, \frac{k_3^2}{k_1^2k_2^2}\prod_{ra}\delta(\pi_r^a - k_r\lambda^a(\rho_r))\,(\frac{-i}{\lambda^1})^3\,\pi_2{}^1(\pi_1{}^1)^2\,\frac{[12]^2[31]}{(\rho_1-\rho_2)(\rho_2-\rho_3)(\rho_3-\rho_1)} \\
&\qquad \times \prod_a \delta(\Sigma_{r=1}^3 k_r\bar{\pi}_{ra})\,\prod_a d\lambda^a\prod_r d\rho_r/d\gamma_S d\gamma_M\, e_{2(1)}e_{2(2)}\bar{C}_{0(3)} \\
&= \delta^4(\Sigma\pi_r\bar{\pi}_r)\,[12]^4\, e_{2(1)}e_{2(2)}\bar{C}_{0(3)}
\end{aligned}
\tag{3.15}
$$

is the antiholomorphic version of (3.6).

Of course, we expect these results for the $d = 0$ amplitudes, from the conjugation properties of the vertex operators. But we present the derivations to demonstrate our computational methods, and to verify (3.7). The $d = 0$ three-graviton coupling vanishes identically, since the vertex operator $e_{-2}(\rho)$ involves $\lambda_a(\rho)\partial\lambda^a(\rho)$ which vanishes for $\lambda^a(\rho) = \lambda^a_0$, a constant:

$$\langle e_2(\rho_1)e_2(\rho_2)e_{-2}(\rho_3)\rangle_{\rm tree} = \int \langle 0|e_2(\rho_1)e_2(\rho_2)e_{-2}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S$$

$$= -\int \prod_{r=1}^{3} dk_r \, \frac{k_3^5}{k_1^2 k_2^2} \langle 0| \prod_{r=1}^{2} e^{ik_r\bar\pi_{rb}\mu^b(\rho_r)} \, \bar\pi_r^{\dot a} Y_{\dot a}(\rho_r) \; e^{ik_3\bar\pi_{r\dot b}\mu^{\dot b}(\rho_3)}|0\rangle \; \prod_r d\rho_r/d\gamma_S d\gamma_M$$

$$\times \langle 0| \prod_{ra} \delta(\pi_r^a - k_r\lambda^a(\rho_r)) \; \lambda_a(\rho_3)\partial\lambda^a(\rho_3)|0\rangle \; e_{2(1)}e_{2(2)}e_{-2(3)} = 0, \qquad (3.16)$$

and

$$\langle \bar C(\rho_1)e_2(\rho_2)C(\rho_3)\rangle_{\rm tree} = \int \langle 0|\bar C(\rho_1)e_2(\rho_2)C(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S$$

$$= -\int \prod_{r=1}^{3} dk_r \, \frac{k_1^2 k_3}{k_2^2} \langle 0| \prod_{r=1}^{2} e^{ik_r\bar\pi_{r\dot b}\mu^{\dot b}(\rho_r)} \, \bar\pi_r^{\dot a} Y_{\dot a}(\rho_r) \; e^{ik_3\bar\pi_{r\dot b}\mu^{\dot b}(\rho_3)}|0\rangle \; \prod_r d\rho_r/d\gamma_S d\gamma_M$$

$$\times \langle 0| \prod_{ra} \delta(\pi_r^a - k_r\lambda^a(\rho_r)) \, \lambda_a(\rho_3)\partial\lambda^a(\rho_3)|0\rangle \; \bar C_{0(1)}e_{2(2)}C_{0(3)} = 0. \qquad (3.17)$$

These are FFG trees. Finally, we include the familiar degree zero three-gluon vertex

$$\langle A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)A_{-1}^{A_3}(\rho_3)\rangle_{\rm tree} = \int \langle 0|A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)A_{-1}^{A_3}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S$$

$$= \int \prod_{r=1}^{3} dk_r \frac{k_3^3}{k_1 k_2} \prod_{ra} \delta(\pi_r^a - k_r\lambda^a) \; \frac{f^{A_1A_2A_3}}{(\rho_1-\rho_2)(\rho_2-\rho_3)(\rho_3-\rho_1)}$$

$$\times \langle 0|e^{\sum_{r=1}^{3} ik_r\bar\pi_{r\dot b}\mu^{\dot b}(\rho_r)}|0\rangle \prod_a d\lambda^a \prod_r d\rho_r/d\gamma_S d\gamma_M \; A_{1(1)}A_{1(2)}A_{-1(3)}$$

$$= \delta^4(\Sigma\pi_r\bar\pi_r) \, f^{A_1A_2A_3} \; \frac{[12]^3}{[23][31]} \, A_{1(1)}A_{1(2)}A_{-1(3)}. \qquad (3.18)$$

$\langle A_1^{A_1}A_1^{A_2}\bar C\rangle = -[12]^2\delta^{A_1A_2}A_{1(1)}A_{1(2)}\bar C_{0(3)}\delta^4(\Sigma\pi_r\bar\pi_r)$
$\langle A_{-1}^{A_1}A_1^{A_2}e_2\rangle = -\frac{[23]^4}{[12]^2}\delta^{A_1A_2}A_{-1(1)}A_{1(2)}e_{2(3)}\delta^4(\Sigma\pi_r\bar\pi_r)$
$\langle e_2e_2\bar C\rangle = [12]^4 e_{2(1)}e_{2(2)}\bar C_{0(3)}\delta^4(\Sigma\pi_r\bar\pi_r)$
$\langle e_2e_2e_{-2}\rangle = 0$
$\langle \bar C e_2 C\rangle = 0$
$\langle A_1^{A_1}A_1^{A_2}A_{-1}^{A_3}\rangle = \frac{[12]^3}{[23][31]}f^{A_1A_2A_3}A_{1(1)}A_{1(2)}A_{-1(3)}\delta^4(\Sigma\pi_r\bar\pi_r)$

Table 5: $d=0$ Unprimed conformal supergravity couplings

3.2 *Amplitudes with Primed Vertices*

In this section we will compute tree amplitudes containing states with primed vertex operators. As a preliminary study, consider the $d = 1$ coupling $\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}}$ with $\phi\phi G'$ vertex operators. Using the previous methods, it is convenient to evaluate the primed coupling as

$$\begin{aligned}
&\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}} = \int \langle 0|e^{q_0} A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= -\int \prod_{r=1}^{3} dk_r \prod_{r=1}^{3} k_r \prod_{ra} \delta({\pi_r}^a - k_r\lambda^a(\rho_r))\, k_1^2 k_2^2\, (\rho_1-\rho_2)^2 \prod_{r=1}^{3} d\rho_r \prod_a d^2\lambda^a/d\gamma_M d\gamma_S \delta^{A_1A_2} \\
&\quad \times \langle 0|e^{q_0}\, i\left(\frac{s_{3\dot a}}{k_3}\partial\mu^{\dot a}(\rho_3) - s_{3\dot a}\mu^{\dot a}(\rho_3)\bar s_{3a}\partial\lambda^a(\rho_3)\right)\, e^{i\sum_{r=1}^{3} k_r\bar\pi_{r\dot b}\mu^{\dot b}(\rho_r)}|0\rangle\;\; A_{-1(1)}A_{-1(2)}C'_{0(3)} \\
&= -\prod_{r=1}^{3} \frac{\pi_r^1}{(\lambda^1(\rho_r))^2}\, \delta\left(\pi_r^2 - \frac{\lambda^2(\rho_r)}{\lambda^1(\rho_r)}\pi_r^1\right)(\rho_1-\rho_2)^2 (\frac{\pi_1^1\pi_2^1}{\lambda^1(\rho_1)\lambda^1(\rho_2)})^2 \\
&\quad\quad \times \prod_{a,\dot a} d^2\lambda^a d^2\mu^{\dot a} \prod_r d\rho_r/d\gamma_S d\gamma_M\; \delta^{A_1A_2}\; A_{-1(1)}A_{-1(2)}C'_{0(3)} \\
&\quad\quad\quad \times i\Big(\frac{s_{31}}{\pi_3^1}\;\left(\lambda_0^1\mu_{-1}^1 - \mu_0^1\lambda_{-1}^1\right) + \frac{s_{32}}{\pi_3^2}\;\left(\lambda_0^2\mu_{-1}^2 - \mu_0^2\lambda_{-1}^2\right)\Big) e^{i\sum_{r=1}^{3}\frac{\pi_r^1}{\lambda^1(\rho_r)}\bar\pi_{r\dot b}(\mu_0^{\dot b}+\rho_r\mu_{-1}^{\dot b})}
\end{aligned} \tag{3.19}$$

where we have used the delta functions $\delta(\pi_r^1 - k_r\lambda^1(\rho_r))$ to do the k_r integrations. Here $\lambda^a(\rho) = \lambda_0^a + \rho\lambda_{-1}^a$. In order to perform the $d^2\mu^{\dot a}$ integrations, we note that

$$\sum_{r=1}^{n} \pi_r^b\bar\pi_{r\dot a} = \sum_{r=1}^{n} \frac{\lambda^b(\rho_r){\pi_r}^1\bar\pi_{r\dot a}}{\lambda^1(\rho_r)} = \lambda_0^b \sum_{r=1}^{n}\frac{\pi_r^1\bar\pi_{r\dot a}}{\lambda^1(\rho_r)} + \lambda_{-1}^b\sum_{r=1}^{n}\frac{\pi_r^1\bar\pi_{r\dot a}\rho_r}{\lambda^1(\rho_r)} \tag{3.20}$$

for any n, when $\pi_r^2 - (\lambda^2(z_r)/\lambda^1(z_r))\pi_r^1 = 0$. We can invert this change of variables to write $\sum_{r=1}^{2}\frac{\pi_r^1}{\lambda^1(\rho_r)}\bar\pi_{r\dot a}$ and $\sum_{r=1}^{2}\frac{\pi_r^1}{\lambda^1(\rho_r)}\bar\pi_{r\dot a}\rho_r$ in terms of $\sum_{r=1}^{2}\pi^b\bar\pi_{\dot a}$, and express the exponential in (3.19) as

$$e^{i\sum_{r=1}^{3}\frac{\pi_r^1}{\lambda^1(\rho_r)}\bar\pi_{r\dot b}(\mu_0^{\dot b}+\rho_r\mu_{-1}^{\dot b})} = e^{i\frac{\epsilon_{ca}}{\det\lambda}\left(\lambda_0^a\mu_{-1}^{\dot b}-\lambda_{-1}^a\mu_0^{\dot b}\right)\sum_{r=1}^{3}\pi_r^c\bar\pi_{r\dot b}}, \tag{3.21}$$

where the anti-symmetric epsilon tensor is $\epsilon^{12} = 1 = -\epsilon_{12}$, as in section 2. Then the integrand of the $d^2\mu^{\dot a}$ integrations can be expressed as derivatives of the exponential,

$$\int \prod_{\dot a} d^2\mu^{\dot a}\Big(\frac{s_{31}}{\pi_3^1}\ \left(\lambda_0^1\mu_{-1}^1-\mu_0^1\lambda_{-1}^1\right)+\frac{s_{32}}{\pi_3^2}\ \left(\lambda_0^2\mu_{-1}^2-\mu_0^2\lambda_{-1}^2\right)\Big)e^{i\sum_{r=1}^3\frac{\pi_r^1}{\lambda^1(\rho_r)}\bar\pi_{r\dot b}(\mu_0^{\dot b}+\rho_r\mu_{-1}^{\dot b})}$$
$$=(-i\det\lambda)\left(\frac{s_{31}}{\pi_3^1}\ \frac{\partial}{\partial\sum_{r=1}^3\pi_r^2\bar\pi_{r1}}-\frac{s_{32}}{\pi_3^2}\ \frac{\partial}{\partial\sum_{r=1}^3\pi_r^1\bar\pi_{r2}}\right)\int\prod_{\dot a} d^2\mu^{\dot a}e^{i\frac{\epsilon_{ca}}{\det\lambda}\left(\lambda_0^a\mu_{-1}^{\dot b}-\lambda_{-1}^a\mu_0^{\dot b}\right)\sum_{r=1}^3\pi_r^c\bar\pi_{r\dot b}}. \tag{3.22}$$

Performing the $d^2\mu^{\dot a}$ integrals to find momentum delta functions,

$$\int\prod_{\dot a} d^2\mu^{\dot a}e^{i\frac{\epsilon_{ca}}{\det\lambda}\left(\lambda_0^a\mu_{-1}^{\dot b}-\lambda_{-1}^a\mu_0^{\dot b}\right)\sum_{r=1}^3\pi_r^c\bar\pi_{r\dot b}}=(\det\lambda)^2\,\delta^4(\Sigma\pi_r\bar\pi_r), \tag{3.23}$$

and using our previous methods, (3.19) becomes

$$\begin{aligned}
&\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\rm tree}\\
&=-\langle 12\rangle^2\Big[\frac{s_{31}}{\pi_3^1}\,\delta^2(\sum_{r=1}^3\pi_r^a\bar\pi_{r2})\,\delta(\sum_{r=1}^3\pi_r^1\bar\pi_{r1})\,\delta'\left(\sum_{r=1}^3\pi_r^2\bar\pi_{r1}\right)\\
&\qquad-\frac{s_{32}}{\pi_3^2}\,\delta^2(\sum_{r=1}^3\pi_r^a\bar\pi_{r1})\ \delta(\sum_{r=1}^3\pi_r^2\bar\pi_{r2})\ \delta'\left(\sum_{r=1}^3\pi_r^1\bar\pi_{r2}\right)\Big]\ \delta^{A_1A_2}A_{-1(1)}A_{-1(2)}C'_{0(3)}\\
&=\langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\frac{C'_{0(3)}}{2p_3^0}\frac{\partial}{\partial P^0}\delta^4(\Sigma\pi_r\bar\pi_r)
\end{aligned} \tag{3.24}$$

where we have chosen the Berkovits Witten gauge $s_{\dot a}=\frac{\pi^a\sigma^0_{a\dot a}}{2p^0}$, so $\frac{s_{21}}{\pi_2^1}=\frac{s_{22}}{\pi_2^2}=\frac{1}{\pi_2^1\bar\pi_2^1+\pi_2^2\bar\pi_2^2}=\frac{1}{2p_2^0}$, and defined $P^0=\sum_{r=1}^3p_r^0=\frac{1}{2}\sum_{r=1}^3(\pi_r^1\bar\pi_{r2}-\pi_r^2\bar\pi_{r1})$, using $p_{ra\dot a}=\pi_{ra}\bar\pi_{r\dot a}=\sigma^\mu_{a\dot a}p_{r\mu}$ as in section 2.

We interpret the amplitude (3.24) with the help of understanding how the momentum operator acts on the primed states. In conformal supergravity, the dipole pairs arise as solutions to equations of motion with higher than quadratic derivatives, see for example [9, 3]. Each pair σ_p,σ'_p satisfies $(\partial_\mu\partial^\mu)^2\sigma=0$, and comprises a plane wave state $\sigma_p=e^{ip\cdot x}$, and a state $\sigma'_p=iA\cdot xe^{ip\cdot x}$ that cannot diagonalize the momentum operator for any non-zero vector A independent of x. Since $P^{\rm op}_{a\dot a}=-i\frac{\partial}{\partial x^{a\dot a}}$, then

$$P^{\rm op}_{a\dot a}\sigma_p=p_{a\dot a}\sigma_p,\qquad P^{\rm op}_{a\dot a}\sigma'_p=p_{a\dot a}\sigma'_p+A_{a\dot a}\sigma_p. \tag{3.25}$$

In particular, we can write $\sigma'_p=A^{a\dot a}\frac{\partial}{\partial p^{a\dot a}}\sigma_p$, and choose A to be in the time direction [3] to

make contact with the Berkovits Witten gauge, so

$$\sigma'_p \sim \frac{\partial}{\partial p^0}\sigma_p. \tag{3.26}$$

The primed amplitude (3.24) is effectively $-\frac{C'_{0(3)}}{2p_3^0 C_{0(3)}}\frac{\partial}{\partial p_3^0}$ times the form (3.2), as expected in view of (3.26). For the pair of states to have the same relative dimension, the wavefunctions $C_{0(r)}, C'_{0(r)}$ differ in dimension by a factor of $(p^0)^2$, so the primed amplitude (3.24) has canonical dimensions.

But what about momentum conservation? Surely primed amplitudes are conformally invariant, just as the others. Although the primed states are not eigenstates of the momentum operator, we know they transform as in (3.25). So, the momentum operator acts on the coupling $\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}}$ as

$$\begin{aligned}
&P^0\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}} - \frac{C'_{0(3)}}{C_{0(3)}2p_3^0}\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C(\rho_3)\rangle_{\text{tree}} \\
&= \langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\frac{C'_{0(3)}}{2p_3^0}\;P^0\frac{\partial}{\partial P^0}\delta(P^0)\delta^3(P^i) \\
&\qquad\qquad + \frac{C'_{0(3)}}{2p_3^0}\langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\delta(P^0)\delta^3(P^i) \\
&= 0
\end{aligned} \tag{3.27}$$

and

$$P^i\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}} = \langle 12\rangle^2\,\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}\;C'_{0(3)}\frac{\partial}{\partial P^0}\delta(P^0)P^i\delta^3(P^i) = 0, \tag{3.28}$$

verifying the primed amplitude (3.24) has translational invariance. Here $P^\mu = \sum_r p_r^\mu$, and $P^0\frac{\partial}{\partial P^0}\delta(P^0) = -\delta(P^0)$ on the support of a test function.

In a similar calculation, now using the $e'_{-2}(\rho)$ vertex operator in lieu of $C'(\rho)$, we find the

MHV coupling for two gluons and a primed graviton:

$$\begin{aligned}
\langle A_1^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)e'_{-2}(\rho_3)\rangle_{\text{tree}} &= \int \langle 0|e^{q_0}A_1^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)e'_{-2}(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
= -\frac{\langle 23\rangle^4}{\langle 12\rangle^2}\delta^{A_1A_2}A_{1(1)}A_{-1(2)}e'_{-2(3)}&\Big[\frac{s_{31}}{\pi_3^1}\,\frac{\partial}{\partial\sum_{r=1}^3\pi_r^2\bar{\pi}_{r1}} - \frac{s_{32}}{\pi_3^2}\,\frac{\partial}{\partial\sum_{r=1}^3\pi_r^1\bar{\pi}_{r2}}\Big]\ \delta^4(\sum_{r=1}^{2}\pi_r\bar{\pi}_r) \\
= \frac{\langle 23\rangle^4}{\langle 12\rangle^2}\delta^{A_1A_2}A_{1(1)}A_{-1(2)}&\frac{e'_{-2(3)}}{2p_3^0}\frac{\partial}{\partial P^0}\delta^4(\Sigma\pi_r\bar{\pi}_r).
\end{aligned} \tag{3.29}$$

For the $d=1$ GGG coupling of two gravitons and a scalar, we can extend to any combination of primed vertices $V_{G'}$ as follows. If there is more than one primed vertex operator, there will be a product of factors in the derivation of the amplitude, of the form,

$$\Big(\frac{s_{r1}}{\pi_r^1}\ \big(\lambda_0^1\mu_{-1}^1 - \mu_0^1\lambda_{-1}^1\big) + \frac{s_{r2}}{\pi_r^2}\ \big(\lambda_0^2\mu_{-1}^2 - \mu_0^2\lambda_{-1}^2\big)\Big) \tag{3.30}$$

for each site r that corresponds to a primed vertex. We can evaluate this in a similar way to (3.22), to find, for example,

$$\begin{aligned}
\langle e'_{-2}(\rho_1)e'_{-2}(\rho_2)C'(\rho_3)\rangle_{\text{tree}} &= \langle 0|e^{q_0}e'_{-2}(\rho_1)e'_{-2}(\rho_2)C'(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_M d\gamma_S \\
&= -\langle 12\rangle^4\,\frac{e'_{-2(1)}}{2p_1^0}\frac{e'_{-2(2)}}{2p_2^0}\frac{C'_{0(3)}}{2p_3^0}\frac{\partial^3}{(\partial P^0)^3}\delta^4(\Sigma\pi_r\bar{\pi}_r).
\end{aligned} \tag{3.31}$$

So effectively, the contribution of a $V_{G'}(\rho_r)$ vertex operator to a tree amplitude can be found by replacing *each* unprimed wavefunction by a primed wavefunction times $-\frac{1}{2p_r^0}\frac{\partial}{\partial p_r^0}$.

Amplitudes involving V'_F vertices are more tedious to evaluate. As a guide for these methods, we can use the antiholomorphic amplitudes. For example, the $d=0$ three-point coupling

$$
\begin{aligned}
&\langle A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)\bar{C}'(\rho_3)\rangle_{\text{tree}} = \int \langle 0|A_1^{A_1}(\rho_1)A_1^{A_2}(\rho_2)\bar{C}'(\rho_3)|0\rangle \prod_{r=1}^{3} d\rho_r/d\gamma_S d\gamma_M \\
&= -\int \prod_{r=1}^{3} dk_r \frac{k_3^2}{k_1 k_2} \prod_{r=1}^{3} d\rho_r/d\gamma_S d\gamma_M \ \left(\frac{\delta^{A_1 A_2} A_{1(1)} A_{1(2)} \bar{C}'_{0(3)}}{(\rho_1-\rho_2)^2} \right) \\
&\quad \times \Big(\ \bar{s}_3^a \ \langle 0| \prod_{c,r=1}^{2} \delta(\pi_r^c - k_r \lambda^c(\rho_r)) \, \delta(\pi_3^c - k_3 \lambda^c(\rho_r)) \ Y_a(\rho_3)|0\rangle \int \prod_a d\mu_0^a e^{i\sum_{r=1}^{3} k_r \bar{\pi}_{rb}\mu_0^b} \\
&\qquad + i s_3^{\dot{a}} \ \langle 0| e^{i\sum_{r=1}^{2} k_r \bar{\pi}_{r\dot{b}} \mu^{\dot{b}}(\rho_r)} e^{ik_3 \bar{\pi}_{3\dot{b}}\mu^{\dot{b}}(\rho_3)} \, Y_{\dot{a}}(\rho_3)|0\rangle \\
&\qquad\qquad \times \int \prod d\lambda_0^a \prod_{c,r=1}^{2} \delta(\pi_r^c - k_r \lambda^c(\rho_r)) \ \bar{s}_3^a \frac{\partial}{\partial \pi_3^a} \delta(\pi_3^c - k_3 \lambda^c(\rho_3)) \Big) \\
&= [12]^2 \, \delta^{A_1 A_2} A_{1(1)} A_{1(2)} \frac{\bar{C}'_{0(3)}}{2p_3^0} \frac{\partial}{\partial P^0} \delta^4(\Sigma \pi_r \bar{\pi}_r). \qquad (3.32)
\end{aligned}
$$

where we could evaluate $\langle 0| \prod_{c,r=1}^{2} \delta(\pi_r^c - k_r\lambda^c(\rho_r))\,\delta(\pi_3^c - k_3\lambda^c(\rho_r)) \ Y_a(\rho_3)|0\rangle$ by writing the delta functions $\prod_{c,r} \delta(\pi_r^c - k_r\lambda^c(\rho_r))$ as $\int \prod_{c,r} d\omega_{rc} e^{i\sum_{r=1}^{2} \omega_{rc}(k_r\lambda^c(\rho_r) - \pi_r^c)}$, using the commutator of $\lambda^a(\rho_r)$ with $Y_a(\rho_3)$, for $r = 1, 2$, and divide by the invariant measure (3.13). But we know the result, since it is the antiholomorphic form of (3.24), found by replacing $\pi_{ra}, \bar{\pi}_{r\dot{b}}$ with their conjugates $\bar{\pi}_{r\dot{a}}, \pi_{rb}$.

The $d = 0$ three-point amplitudes from the FGG', $F'GG$, GGG', $FG'G'$, $F'GG'$, $GG'G'$, $F'G'G'$ and $G'G'G'$ sectors all vanish. For $d = 0$, the three-point tree amplitudes from the FFG' and FGF' sectors give conventional Einstein couplings, along the lines of [23]. Three-point trees in the $FF'G'$, $F'F'G$ and $F'F'G'$ sectors for $d = 0$ are more detailed to access.

The MHV three-point amplitudes involving the primed states of the dipoles in the $\phi\phi G'$, GGG', $GG'G'$ and $G'G'G'$ sectors, together with their helicity conjugates are summarized in Tables 6 and 7.

$\langle A_{-1}^{A_1} A_{-1}^{A_2} C' \rangle = \langle 12\rangle^2 \, \delta^{A_1 A_2} A_{-1(1)} A_{-1(2)} \frac{C'_{0(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle A_{1}^{A_1} A_{-1}^{A_2} e'_{-2} \rangle = \frac{\langle 23\rangle^4}{\langle 12\rangle^2} \delta^{A_1 A_2} A_{1(1)} A_{-1(2)} \frac{e'_{-2(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_{-2} e_{-2} C' \rangle = -\langle 12\rangle^4 e_{-2(1)} e_{-2(2)} \frac{C'_{0(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_{-2} e'_{-2} C \rangle = -\langle 12\rangle^4 e_{-2(1)} \frac{e'_{-2(2)}}{2p_2^0} C_{0(3)} \frac{\partial}{\partial p_2^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_{-2} e'_{-2} C' \rangle = -\langle 12\rangle^4 e_{-2(1)} \frac{e'_{-2(2)}}{2p_2^0} \frac{C'_{0(3)}}{2p_3^0} \frac{\partial^2}{\partial p_2^0 \partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e'_{-2} e'_{-2} C \rangle = -\langle 12\rangle^4 \frac{e'_{-2(1)}}{2p_1^0} \frac{e'_{-2(2)}}{2p_2^0} C_{0(3)} \frac{\partial^2}{\partial p_1^0 \partial p_2^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e'_{-2} e'_{-2} C' \rangle = -\langle 12\rangle^4 \frac{e'_{-2(1)}}{2p_1^0} \frac{e'_{-2(2)}}{2p_2^0} \frac{C'_{0(3)}}{2p_3^0} \frac{\partial^3}{\partial p_1^0 \partial p_2^0 \partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$

Table 6: MHV Conformal supergravity couplings with primed states

$\langle A_{1}^{A_1} A_{1}^{A_2} \bar{C}' \rangle = [12]^2 \, \delta^{A_1 A_2} A_{1(1)} A_{1(2)} \frac{\bar{C}'_{0(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle A_{-1}^{A_1} A_{1}^{A_2} e'_{2} \rangle = \frac{[23]^4}{[12]^2} \delta^{A_1 A_2} A_{-1(1)} A_{1(2)} \frac{e'_{2(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_{2} e_{2} \bar{C}' \rangle = -[12]^4 e_{2(1)} e_{2(2)} \frac{\bar{C}'_{0(3)}}{2p_3^0} \frac{\partial}{\partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_{2} e'_{2} \bar{C} \rangle = -[12]^4 e_{2(1)} \frac{e'_{2(2)}}{2p_2^0} \bar{C}_{0(3)} \frac{\partial}{\partial p_2^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e_{2} e'_{2} \bar{C}' \rangle = -[12]^4 e_{2(1)} \frac{e'_{2(2)}}{2p_2^0} \frac{\bar{C}'_{0(3)}}{2p_3^0} \frac{\partial^2}{\partial p_2^0 \partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e'_{2} e'_{2} \bar{C} \rangle = -[12]^4 \frac{e'_{2(1)}}{2p_1^0} \frac{e'_{2(2)}}{2p_2^0} \bar{C}_{0(3)} \frac{\partial^2}{\partial p_1^0 \partial p_2^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$
$\langle e'_{2} e'_{2} \bar{C}' \rangle = -[12]^4 \frac{e'_{2(1)}}{2p_1^0} \frac{e'_{2(2)}}{2p_2^0} \frac{\bar{C}'_{0(3)}}{2p_3^0} \frac{\partial^3}{\partial p_1^0 \partial p_2^0 \partial p_3^0} \delta^4(\Sigma \pi_r \bar{\pi}_r)$

Table 7: $d = 0$ Conformal supergravity couplings with primed states

4 Canonical Derivation of the Berkovits Witten Amplitudes

In this section, we extend our analysis of the three-point functions using canonical quantization, to N-point MHV tree amplitudes for unprimed vertex operators. The maximal helicity violating (MHV) amplitudes contain any two vertex operators of negative helicity, $e_{-2}, \bar{C}, A_{-1}$, and $N-2$ positive helicity vertex operators from the set e_2, C, A_1.

We will compute an amplitude for a specific choice of the two negative helicity states, and then discuss how this generalizes. We consider the $(d=1)$ amplitude for two negative helicity vertex operators, one of type G and one of type F, $\langle e_{-2}\bar{C}e_2 \ldots e_2 C \ldots C A_1 \ldots A_1\rangle_{\text{tree}}$. This has n positive helicity type F gravitons, m positive helicity type G scalars, and p positive helicity gluons . We denote the total number of vertices as $N = 2+n+m+p$. Inserting the expressions from Table 3, we find

$$\begin{aligned}\int \langle 0|e^{q_0} \int dk_1 k_1 \lambda_{a_1}(\rho_1) \prod_{a=1}^{2} \delta(k_1\lambda^a(\rho_1) - \pi_1{}^a) e^{ik_1\bar{\pi}_{1\dot{b}}\mu^{\dot{b}}(\rho_1)} k_1^4 \psi^1(\rho_1)\psi^2(\rho_1)\psi^3(\rho_1)\psi^4(\rho_1)\partial\lambda^{a_1}(\rho_1) \\ \times i\int \frac{dk_2}{k_2^2}\bar{\pi}_2{}^{\dot{a}} \prod_{a=1}^{2} \delta(k_2\lambda^a(\rho_2) - \pi_2{}^a) e^{ik_2\bar{\pi}_{2\dot{b}}\mu^{\dot{b}}(\rho_2)} k_2^4 \psi^1(\rho_2)\psi^2(\rho_2)\psi^3(\rho_2)\psi^4(\rho_2) Y_{\dot{a}}(\rho_2) \\ \times \prod_{j=3}^{n+2} i\int \frac{dk_j}{k_j^2}\bar{\pi}_j{}^{\dot{a}} \prod_{a=1}^{2} \delta(k_j\lambda^a(\rho_j) - \pi_j{}^a) e^{ik_j\bar{\pi}_{j\dot{b}}\mu^{\dot{b}}(\rho_j)} Y_{\dot{a}}(\rho_j) \\ \times \prod_{j=n+3}^{m+n+2} \int dk_j k_j \lambda_{a_j}(\rho_j) \prod_{a=1}^{2} \delta(k_j\lambda^a(\rho_j) - \pi_j^a) e^{ik_j\bar{\pi}_{j\dot{b}}\mu^{\dot{b}}(\rho_j)} \partial\lambda^{a_j}(\rho_j) \\ \times \prod_{j=m+n+3}^{N} \int \frac{dk_j}{k_j} \prod_{a=1}^{2} \delta(k_j\lambda^a(\rho_j) - \pi_j^a) e^{ik_j\bar{\pi}_{j\dot{b}}\mu^{\dot{b}}(\rho_j)} J^{A_j}(\rho_j)|0\rangle \prod_{r=1}^{N} d\rho_r/d\gamma_S d\gamma_M\end{aligned} \tag{4.1}$$

where we have dropped the polarizations for convenience. It is useful to introduce the sets of indices: $\mathbf{n} = \{3, \ldots, n+2\}$, $\mathbf{m} = \{n+3, \ldots, m+n+2\}$, and $\mathbf{p} = \{m+n+3, \ldots, N\}$. To further emphasize the occurrence of gluon or graviton type, we define the larger sets $\mathbf{n}' = \{2, 3, \ldots, n+2\}$ and $\mathbf{m}' = \{1, n+3, \ldots, m+n+2\}$. From the following formula presented below, we can see these sets will be useful when considering amplitudes having a more complicated ordering of vertex operators. We rewrite (4.1) as

$$
\begin{aligned}
&(i)^{n+1}\int\prod_{r=1}^{N}dk_r d\rho_r/d\gamma_S d\gamma_M\prod_{a,r}\delta(k_r\lambda^a(\rho_r)-\pi_r{}^a)\prod_{j\in\mathbf{n}'}\left(\frac{1}{k_j}\right)^2\prod_{j\in\mathbf{m}'}(k_j)\prod_{j\in\mathbf{p}}\left(\frac{1}{k_j}\right)\\
&\quad\times(k_1k_2)^4\langle 0|e^{q_0}\psi^1(\rho_1)\psi^1(\rho_2)\psi^2(\rho_1)\psi^2(\rho_2)\psi^3(\rho_1)\psi^3(\rho_2)\psi^4(\rho_1)\psi^4(\rho_2)|0\rangle\\
&\quad\times\langle 0|e^{q_0}\prod_{j\in\mathbf{m}'}\lambda_a(\rho_j)\partial\lambda^a(\rho_j)|0\rangle\ \langle 0|\prod_{j\in\mathbf{p}}J^{A_j}(\rho_j)|0\rangle\\
&\quad\times\langle 0|e^{q_0}e^{ik_1\bar\pi_{1\dot b}\mu^{\dot b}(\rho_1)}e^{ik_2\bar\pi_{2\dot b}\mu^{\dot b}(\rho_2)}\bar\pi_2{}^{\dot a}Y_{\dot a}(\rho_2)\prod_{j\in\mathbf{n}}\left(e^{ik_j\bar\pi_{j\dot b}\mu^{\dot b}(\rho_j)}\bar\pi_j^{\dot a}Y_{\dot a}(\rho_j)\right)\\
&\qquad\qquad\times\prod_{j\in\mathbf{m}}\left(e^{ik_j\bar\pi_{j\dot b}\mu^{\dot b}(\rho_j)}\right)\prod_{j\in\mathbf{p}}\left(e^{ik_j\bar\pi_{j\dot b}\mu^{\dot b}(\rho_j)}\right)|0\rangle. \qquad (4.2)
\end{aligned}
$$

Many simplifications happen at this stage. With

$$
\langle 0|e^{q_0}\psi^1(\rho_1)\psi^1(\rho_2)|0\rangle=(\rho_1-\rho_2)\langle 0|e^{q_0}\psi^1_{-1}\psi^1_0|0\rangle=(\rho_1-\rho_2), \qquad (4.3)
$$

four factors of $\rho_1-\rho_2$ come from the second line. Evaluating the λ term, we find

$$
\langle 0|e^{q_0}\prod_{j\in\mathbf{m}'}\lambda_{a_j}(\rho_j)\partial\lambda^{a_j}(\rho_j)|0\rangle=\int\prod_a d^2\lambda^a(\det\lambda)^{m+1}, \qquad (4.4)
$$

where $\det\lambda=\lambda^1_0\lambda^2_{-1}-\lambda^2_0\lambda^1_{-1}$, as in section 3.. We use a current algebra contribution [24]

$$
\langle 0|\prod_{j\in\mathbf{p}}J^{A_j}(\rho_j)|0\rangle=f^{A_{m+n+3}\cdots A_N}\prod_{j\in\mathbf{p}}\frac{1}{\rho_j-\rho_{j+1}}, \qquad (4.5)
$$

with $\rho_{N+1}\equiv\rho_{m+n+3}$. In what follows, we denote $f^{A_{m+n+3}\dots A_N}=f^{A\dots A}$. This is merely simplification of notation, as the group indices add no new information not contained in the denominator. We note, for computing MHV amplitudes containing negative helicity gluons, the form (4.5) remains the same with the set $\mathbf{p}$ replaced by the total set of gluons $\mathbf{p}'$.

The last expectation value in (4.2) is equal to

$$
(-i)^{n+1}(\det\lambda)^2\delta^4(\Sigma\pi_r\bar\pi_r)\prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^{N}k_y\frac{[xy]}{\rho_y-\rho_x}, \qquad (4.6)
$$

where now $\delta^4(\Sigma\pi_r\bar\pi_r)\equiv\prod_{\dot a,b}\delta\left(\Sigma_{r=1}^N\pi_r^b\bar\pi_{r\dot a}\right)$. We integrate the k_r's using $\delta(k_r-\pi_r^1/\lambda^1(\rho_r))$,

and evaluate the amplitude (4.2) to obtain

$$\delta^4(\Sigma\pi_r\bar{\pi}_r)\int\prod_{r=1}^{N}d\rho_r\prod_a d^2\lambda^a/d\gamma_S\gamma_M\prod_{r=1}^{N}\delta({\pi_r}^2-\frac{\lambda^2(\rho_r)}{\lambda^1(\rho_r)}{\pi_r}^1)\prod_{r=1}^{N}\frac{1}{\lambda^1(\rho_r)}$$
$$\times\left(\frac{{\pi_1}^1}{\lambda^1(\rho_1)}\frac{{\pi_2}^1}{\lambda^1(\rho_2)}\right)^4(\rho_1-\rho_2)^4\prod_{j\in\mathbf{n}'}\left(\frac{\lambda^1(\rho_j)}{\pi^1{}_j}\right)^2\prod_{j\in\mathbf{m}'}\left(\frac{{\pi_j}^1}{\lambda^1(\rho_j)}\right)\prod_{j\in\mathbf{p}}\left(\frac{\lambda^1(\rho_j)}{{\pi_j}^1}\right)$$
$$\times f^{A\cdots A}\prod_{j\in\mathbf{p}}\left(\frac{1}{\rho_j-\rho_{j+1}}\right)(\det\lambda)^{m+3}\prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^{N}\frac{{\pi_y}^1}{\lambda^1(\rho_y)}\frac{[xy]}{\rho_y-\rho_x} \tag{4.7}$$

We define $\zeta_r=\frac{\lambda^2(\rho_r)}{\lambda^1(\rho_r)}$ and change variables from ρ_r to ζ_r. The identification

$$\sum_{y=1,y\neq x}^{N}\frac{{\pi_y}^1}{\lambda^1(\rho_y)}\frac{[xy]}{\rho_y-\rho_x}=\frac{\det\lambda}{(\lambda^1(\rho_x))^3}\sum_{y=1,y\neq x}^{N}\frac{{\pi_y}^1[xy]}{\zeta_y-\zeta_x} \tag{4.8}$$

follows from

$$\sum_{y\neq x}\frac{\pi_y^1[xy]}{\lambda^1(\rho_y)(\rho_y-\rho_x)}=\frac{1}{\lambda^1(\rho_x)}\sum_{y\neq x}\frac{\pi_y^1[xy]}{(\rho_y-\rho_x)}, \tag{4.9}$$

using $\sum_y\frac{\pi_y^1\bar{\pi}_{y\dot{b}}}{\lambda^1(\rho_y)}=0$, which is provided by the factor $\delta^4(\Sigma\pi_r\bar{\pi}_r)$ in (4.7), in view of the equality (3.20). To implement the change of variables, we have $\zeta_r-\zeta_j=\frac{(\rho_r-\rho_j)\det\lambda}{\lambda^1(\rho_r)\lambda^1(\rho_j)}$, $d\zeta=\frac{\det\lambda}{\lambda^1(\rho)^2}d\rho$, so (4.7) is

$$\delta^4(\Sigma\pi_r\bar{\pi}_r)\int\prod_{r=1}^{N}d\zeta_r\prod_a d^2\lambda^a/d\gamma_Sd\gamma_M\,(\det\lambda)^{-2}\prod_{r=1}^{N}\delta({\pi_r}^2-\zeta_r\pi_r^1)\left({\pi_1}^1{\pi_2}^1(\zeta_1-\zeta_2)\right)^4$$
$$\times\prod_{j\in\mathbf{n}'}\left(\frac{1}{{\pi_j}^1}\right)^2\prod_{j\in\mathbf{m}'}({\pi_j}^1)\prod_{j\in\mathbf{p}}\left(\frac{1}{{\pi_j}^1}\right)f^{A\cdots A}\prod_{j\in\mathbf{p}}\left(\frac{1}{\zeta_j-\zeta_{j+1}}\right)\prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^{N}{\pi_y}^1\frac{[xy]}{\zeta_y-\zeta_x} \tag{4.10}$$

We identify $d\gamma_Sd\gamma_M=d^2\lambda^a(\det\lambda)^{-2}$ and do the ζ_r integrations. Since

$$\prod_{x\in\mathbf{n}'}\left(\frac{1}{{\pi_x}^1}\right)^3\sum_{y=1,y\neq x}^{N}{\pi_y}^1\frac{[xy]}{\zeta_y-\zeta_x}=\prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^{N}\frac{(\pi_y^1)^2}{(\pi_x^1)^2}\frac{[xy]}{\langle xy\rangle}, \tag{4.11}$$

(4.11) can be reexpressed as [3]

$$\prod_{x\in\mathbf{n}'}\sum_{y=1,y\neq x}^{N}\frac{\langle y\xi\rangle^2}{\langle x\xi\rangle^2}\frac{[xy]}{\langle xy\rangle}, \tag{4.12}$$

which is independent of π_ξ,$\bar{\pi}_\xi$, to obtain the result

$$\langle e_{-2}\bar{C}e_2 \cdots e_2 C \cdots C A_1 \dots A_1\rangle$$
$$= \delta^4(\Sigma \pi_r \bar{\pi}_r)\langle 12\rangle^4 f^{A\cdots A} \prod_{j\in\mathbf{p}} \frac{1}{\langle j, j+1\rangle} \prod_{i\in\mathbf{n}'} \sum_{j=1,j\neq i}^{N} \frac{\langle j\xi\rangle^2}{\langle i\xi\rangle^2} \frac{[ij]}{\langle ij\rangle}. \quad (4.13)$$

The amplitude is independent of the order of the positive helicity states, as this corresponds merely to changing the position of the $Y_{\dot{a}}$ fields, and does not affect (4.6). To generalize our expression for any two negative helicity states, it is useful to identify pieces common to all amplitudes: the two negative helicity states in any position ρ_r, ρ_s will give the factor of $\langle rs\rangle^4$, all gluon vertices contribute to $f^{A\cdots A}\prod_{j\in\mathbf{p}'} \frac{1}{\langle j,j+1\rangle}$, defined in (4.5), and the type F vertex operators contribute to the product of sums. The type G vertex operators provide factors of $\det \lambda$, and leave no further mark on the amplitude. Our answer (4.13) thus becomes the Berkovits Witten formula [3], which they found from a path integral formulation, and where we have absorbed a factor $(-i)^F$ in the definition of the vertex operators V_F.

Comparison of Conformal Gravity with Einstein Gravity Amplitudes

To visualize conformal gravity amplitudes better, we use (4.13) to study the conformal four-graviton tree amplitude

$$\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)e_2(\rho_4)\rangle_{CG} = \langle 12\rangle^4 \prod_{j=3,4}\sum_{k\neq j} \frac{[jk]\langle k\xi\rangle^2}{\langle jk\rangle\langle j\xi\rangle^2}$$
$$= -\frac{\langle 12\rangle^4[32]\langle 21\rangle\left(\langle 43\rangle\langle 21\rangle - \langle 23\rangle\langle 41\rangle\right)\ [42]\langle 21\rangle\left(\langle 34\rangle\langle 21\rangle - \langle 24\rangle\langle 31\rangle\right)}{\langle 31\rangle^2\langle 41\rangle^2\langle 34\rangle^2\langle 23\rangle\langle 42\rangle} \qquad (\text{choose } \xi = 1)$$
$$= \frac{\langle 12\rangle^4[34]^4}{(s_{12})^2} \qquad \text{using the identity } \langle 43\rangle\langle 21\rangle - \langle 23\rangle\langle 41\rangle = \langle 13\rangle\langle 24\rangle$$
$$= \frac{s_{23}s_{24}}{s_{12}}\ \langle e_{-2}(1)e_{-2}(2)e_2(3)e_2(4)\rangle_{Einstein}, \quad (4.14)$$

which has fewer poles than Einstein gravity, since the Berends Giele Kuijf expression [14] for Einstein gravity tree amplitudes as a product of Yang-Mills trees is

$$\langle e_{-2}(1)e_{-2}(2)e_2(3)e_2(4)\rangle_{Einstein} = s_{12}\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\ \frac{\langle 12\rangle^3}{\langle 24\rangle\langle 43\rangle\langle 31\rangle}$$
$$= \frac{1}{s_{12}s_{23}s_{24}}\langle 12\rangle^4[34]^4.$$

We can reproduce the conformal gravity four-point function (4.14) from tree level exchange of the scalar field C

$$\langle e_{-2}(\rho_1)e_{-2}(\rho_2)e_2(\rho_3)e_2(\rho_4)\rangle_{CG} = \langle 12\rangle^4 \frac{1}{(s_{12})^2}[34]^4 \tag{4.15}$$

corresponding to the product of the three-point trees $\langle e_{-2}(\rho_1)e_{-2}(\rho_2)C(\rho)\rangle = \langle 12\rangle^4$ and $\langle \bar{C}(\rho)e_2(\rho_3)e_2(\rho_4)\rangle = [34]^4$, times the conformal propagator $\frac{1}{(p^2)^2}$, where $p^2 = s_{12}$.

5 N-point Tree Amplitudes for Mixed Primed and Unprimed Vertices

Finally we turn to the N-point MHV scattering amplitudes containing both primed and unprimed vertices. To begin, consider two negative helicity gluons, A_{-1} and n G' scalars, C'_0. The total number of vertices is $N = 2+n$. The set of primed vertices is $\mathbf{n} = \{3,\dots,N\}$. To compute $\langle A^{A_1}_{-1}(\rho_1)A^{A_2}_{-1}(\rho_2)C'_0(\rho_3)\dots C'_0(\rho_N)\rangle_{\text{tree}}$, use the vertex operators from Table 3,

$$\begin{aligned}
&\int \langle 0|e^{q_0}\int \frac{dk_1}{k_1}\prod_a \delta\left(k_1\lambda^a(\rho_1)-\pi_1{}^a\right)e^{ik_1\bar{\pi}_{1\dot{b}}\mu^{\dot{b}}(\rho_1)}k_1^4\psi^1(\rho_1)\psi^2(\rho_1)\psi^3(\rho_1)\psi^4(\rho_1)A_{-1(1)}J^{A_1}(\rho_1)\\
&\quad\times\int \frac{dk_2}{k_2}\prod_a \delta\left(k_2\lambda^a(\rho_2)-\pi_2{}^a\right)e^{ik_2\bar{\pi}_{2\dot{b}}\mu^{\dot{b}}(\rho_2)}k_2^4\psi^1(\rho_2)\psi^2(\rho_2)\psi^3(\rho_2)\psi^4(\rho_2)A_{-1(2)}J^{A_2}(\rho_2)\\
&\quad\times\prod_{j\in\boldsymbol{n}}\Big(i\int dk_jk_j\prod_a \delta\left(k_j\lambda^a(\rho_j)-\pi^a{}_j\right)e^{ik_j\bar{\pi}_{j\dot{b}}\mu^{\dot{b}}(\rho_j)}\\
&\qquad\qquad\times\left[s_{j\dot{a}}\partial\mu^{\dot{a}}(\rho_j)-\bar{s}_{ja}s_{j\dot{a}}\mu^{\dot{a}}(\rho_j)\partial\lambda^a(\rho_j)\right]C'_{0(j)}\Big)|0\rangle\prod_{r=1}^N d\rho_r/d\gamma_S d\gamma_M,
\end{aligned}\tag{5.1}$$

which yields

$$\begin{aligned}
&i^n\int\prod_{r=1}^N dk_rd\rho_rk_r\prod_{a,r}\delta\left(k_r\lambda^a(\rho_r)-\pi_r{}^a\right)(\rho_1-\rho_2)^4(k_1k_2)^4\prod_a d^2\lambda^a/d\gamma_Sd\gamma_M\\
&\quad\times\frac{-\delta^{A_1A_2}}{(\rho_1-\rho_2)^2}\left(\frac{1}{k_1k_2}\right)^2A_{-1(1)}A_{-1(2)}C'_{0(3)}\cdots C'_{0(N)}\\
&\quad\times\langle 0|e^{q_0}\prod_{j\in\boldsymbol{n}}\left(\frac{s_{j\dot{a}}}{k_j}\partial\mu^{\dot{a}}(\rho_j)-\bar{s}_{ja}s_{j\dot{a}}\mu^{\dot{a}}(\rho_j)\partial\lambda^a(\rho_j)\right)e^{i\sum_r k_r\bar{\pi}_{r\dot{b}}\mu^{\dot{b}}(\rho_j)}|0\rangle.
\end{aligned}\tag{5.2}$$

We evalute the expectation value in (5.2) as a sequence of derivatives, as in (3.31), and find

$$
\begin{aligned}
&\langle A_{-1}^{A_1}(\rho_1)A_{-1}^{A_2}(\rho_2)C_0'(\rho_3)\dots C_0'(\rho_N)\rangle_{\rm tree} \\
&= -\langle 12\rangle^2\delta^{A_1A_2}A_{-1(1)}A_{-1(2)}C_{0(3)}'\dots C_{0(N)}'\prod_{j\in\boldsymbol{n}}\left[-\frac{1}{2p_j^0}\frac{\partial}{\partial p_j^0}\right]\delta^4(\Sigma\pi_r\bar{\pi}_r). \qquad (5.3)
\end{aligned}
$$

Clearly this same form holds for

$$
\begin{aligned}
&\langle e_{-2}'(\rho_1)e_{-2}'(\rho_2)C_0'(\rho_3)\dots C_0'(\rho_N)\rangle_{\rm tree} \\
&= \langle 12\rangle^4 e_{-2(1)}'e_{-2(2)}'C_{0(3)}'\dots C_{0(N)}'\prod_{j\in N}\left[-\frac{1}{2p_j^0}\frac{\partial}{\partial p_j^0}\right]\delta^4(\Sigma\pi_r\bar{\pi}_r), \qquad (5.4)
\end{aligned}
$$

and for any combination of these type G primed and unprimed states, with the product then taken over the primed sites.

N-point functions with type F' vertices, and with mixed G' and F are more varied to track.

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